

GENERALISED TRIANGLE GROUPS OF TYPE (2,3,2) WITH NO CYCLIC ESSENTIAL REPRESENTATIONS

JAMES HOWIE AND ALEXANDER KONOVALOV

1. BACKGROUND

A *generalised triangle group* is a group G given by a presentation of the form

$$\langle x, y | x^p = y^q = W(x, y)^r = 1 \rangle,$$

where $p, q, r > 1$ are integers and W is a cyclically reduced word in the free product

$$\langle x, y | x^p = y^q = 1 \rangle \cong \mathbb{Z}_p * \mathbb{Z}_q$$

of even syllable length $2\ell > 0$. We say that G is of type (p, q, r) and has *length parameter* ℓ .

A conjecture of Rosenberger [8] says that this class of groups satisfies a Tits alternative: either G is virtually soluble, or G contains a non-abelian free subgroup. Known results to date (see for example the summary in [5]) reduce the conjecture to the three cases $(p, q, r) \in \{(2, 3, 2), (2, 4, 2), (2, 5, 2)\}$.

An *essential representation* from G to a group H is a homomorphism $\phi : G \rightarrow H$ such that $\phi(x)$, $\phi(y)$ and $\phi(W)$ have orders p, q, r respectively. In this paper we concentrate on the case where $(p, q, r) = (2, 3, 2)$ and G does not admit an essential representation onto a cyclic group. For a generalised triangle group of type $(2, 3, 2)$, an essential representation onto a cyclic group exists if and only if the exponent sums of x and y in W are respectively odd and divisible by 3. Hence the absence of such a representation can arise in one of two ways: either G has even length parameter, or it has odd length parameter and the exponent-sum of y in W is not divisible by 3.

When G has even length parameter, the conjecture was proved modulo six outstanding cases in [4]. Jack Button (private communication) has since been able to use his largeness-testing software [1] to prove the conjecture for four of these six. In Appendix A below we include logs of GAP and MAPLE sessions which confirm Buttons largeness results for these four groups, as well as verifying the Rosenberger conjecture for one of the remaining two groups. The only remaining open case of the six exceptions in [4] is

$$(1) \quad \langle x, y | x^2 = y^3 = ((xy)^4(xy^2)^3(xy)^2xy^2)^2 = 1 \rangle.$$

The argument in [4] involved finding an upper bound for ℓ ($\ell \leq 40$) in any putative counterexample, and systematically analysing the finite (but large) set of words no longer than that bound.

Date: December 2, 2016.

In the current note we concentrate mainly on the case where $(p, q, r) = (2, 3, 2)$, ℓ is odd and the exponent-sum of y in W is not divisible by 3. Again we achieve this by finding a bound on ℓ (in this case $\ell \leq 49$) and analysing (using parallel computations with GAP [2]) the resulting finite set of words.

We show that, with at most one exception (up to isomorphism), the Rosenberger Conjecture holds for all such groups. The exception is the group

$$(2) \quad \langle x, y | x^2 = y^3 = ((xy)^4(xy^2)^3(xy)^2(xy^2)^2)^2 = 1 \rangle.$$

We have not been able to prove the conjecture for this group, but there is no evidence to suggest that it fails.

Unfortunately, this approach does not apply to the case where $(p, q, r) = (2, 3, 2)$ and G admits an essential cyclic representation, in other words ℓ is odd and the exponent-sum of y in W is divisible by 3: we cannot find a theoretical upper bound for ℓ in that case.

Summarizing the above discussion, we have

Theorem 1. *Let G be a generalised triangle group of type $(2, 3, 2)$ that does not admit an essential representation onto a cyclic group. If G is a counterexample to the Rosenberger conjecture, then it is isomorphic to one of the following:*

$$\langle x, y | x^2 = y^3 = ((xy)^4(xy^2)^3(xy)^2xy^2)^2 = 1 \rangle,$$

$$\langle x, y | x^2 = y^3 = ((xy)^4(xy^2)^3(xy)^2(xy^2)^2)^2 = 1 \rangle.$$

In §2 below we treat the even length case. We outline the proof that 5 of the 6 exceptional cases from [4, Table 2] contain non-abelian free subgroups. The full computational details are postponed to Appendix A. In the remainder of the paper we concentrate on the odd-length case.

In §3 we recall the definition of the *trace polynomial* of W , and use it to obtain strong restrictions on the form of W in any putative counterexample to the Rosenberger Conjecture (Theorem 4). In §4 we describe a computer search that uses these restrictions to refine the set of possible counterexamples (up to a simple equivalence relation) to a set of size 31. In §5 we use small-cancellation theory to eliminate 16 of these groups as potential counterexamples. A more complicated application of small-cancellation techniques is used to eliminate one further group – stated as Theorem 6 in §6 and proved in Appendix C. Finally in §7 we use a variety of further methods to eliminate 13 more groups, leaving us with the single exceptional case shown above.

Our work involves a certain amount of computation. The graphs in the proof of Theorem 4 were produced using MAPLE, which was also used for the Gröbner basis calculations in Appendix A. All other computations were done using GAP [2]. Logs of GAP and MAPLE sessions verifying most of our computational results are included in Appendices A (for the even length case) and B (for the odd length case). Appendix C contains a detailed proof using pictures of the existence of free subgroups in one of the groups concerned.

2. EVEN LENGTH

In this section we outline the investigations into the case where ℓ is even. Detailed calculations are given in Appendix A. As mentioned in the Introduction, [4] deals with all such groups with 6 exceptions. Button has used his largeness test [1] to deal with 4 of the 6 exceptions and we confirm this by GAP calculations in Appendix A. We are currently unable to handle the fifth exceptional case (1) above. However we are able to confirm that the sixth exception (Group 7b in [4, Table 2]) has non-abelian free subgroups.

Theorem 2. *The group*

$$G = \langle x, y | x^2 = y^3 = ((xy)^3(xy^2)^2xy(xy^2)^2xyxy^2)^2 = 1 \rangle$$

contains non-abelian free subgroups.

Proof. The commutator subgroup $[G, G]$ of G has index 6 and a presentation with two generators u, v and six relators:

$$\begin{aligned} W_1^2 &= W_1(u, v)^2 = (v^2uv^{-1}u^2)^2, \\ W_2^2 &= W_2(u, v)^2 = (v^2u^2v^{-1}u)^2, \\ W_3^2 &= W_3(u, v)^2 = (v^2u^{-3}vu^{-1})^2, \\ W_4^2 &= W_4(u, v)^2 = (v^2u^{-1}vu^{-3})^2, \\ W_5^2 &= W_5(u, v)^2 = (v^2u^{-1}vu^{-1}vu)^2, \\ W_6^2 &= W_6(u, v)^2 = (v^2uvu^{-1}vu^{-1})^2. \end{aligned}$$

A Gröbner basis calculation shows that $[G, G]$ has a representation $\sigma : [G, G] \rightarrow PSL(2, \mathbb{C})$, $u \mapsto M$, $v \mapsto N$, where

$$M = \begin{pmatrix} 0 & -1 \\ 1 & a \end{pmatrix}, \quad N = \begin{pmatrix} -z & 1 + bz + z^2 \\ -1 & z + b \end{pmatrix},$$

z is an algebraic integer of degree 48 and $a = \text{Trace}(M)$ and $b = \text{Trace}(N)$ are roots in $\mathbb{Q}[z]$ of the irreducible polynomial

$$p(X) := X^{12} - 3X^{10} - 9X^8 + 42X^6 - 48X^4 + 15X^2 + 1.$$

Now a representation $\rho : \langle u, v \rangle \rightarrow PSL(2, \mathbb{C})$ is reducible only if

$$\text{Trace}(\rho(u))^2 + \text{Trace}(\rho(v))^2 + \text{Trace}(\rho(uv))^2 + \text{Trace}(\rho(u))\text{Trace}(\rho(v))\text{Trace}(\rho(uv)) = 4.$$

A further Gröbner basis calculation shows that

$$a^2 + b^2 + c^2 + abc \neq 4 \text{ in } \mathbb{Q}[z],$$

where $c = \text{Trace}(MN)$. Hence the above representation σ is irreducible. Moreover, since p is irreducible of degree 12, none of a, b, c can be the trace of a matrix of order less than 25 in $SL(2, \mathbb{C})$, so the image of σ in $PSL(2, \mathbb{C})$ cannot be a finite or dihedral subgroup. In particular $\sigma([G, G])$, and hence G , contains a non-abelian free subgroup.

Logs of sessions in GAP and MAPLE carrying out the calculations listed above are given in Appendix A below. \square

3. TRACE POLYNOMIAL

The standard tool for investigating generalised triangle groups is the trace polynomial, defined as follows. Construct matrices $X, Y \in \mathrm{SL}_2(\mathbb{C}[\lambda])$ such that the traces of X, Y, XY are respectively equal to $2\cos(\pi/p)$, $2\cos(\pi/q)$ and λ . The trace polynomial $\tau_W(\lambda)$ is defined to be the trace of $W(X, Y)$. It is not difficult to verify that such a choice of matrices is always possible, and that τ_W does not depend on the choice. Moreover, $\tau_W(\lambda)$ is a polynomial of degree k in λ , and the combinatorics of the word W determine the coefficients of τ_W in a well-understood way. Details can be found, for example, in [6].

If $\alpha \in \mathbb{C}$ is a root of $\tau_W(\lambda) - 2\cos(\pi/r)$, then we can obtain an *essential representation* of G (one which maps x, y, w to elements of orders p, q, r respectively) into $\mathrm{PSL}_2(\mathbb{C})$ by putting $\lambda = \alpha$ in X, Y . Provided the image of this representation is non-elementary, it will contain a non-abelian free subgroup by Tits' results, and hence so will G . Thus one is reduced to consideration of words W such that the roots of $\tau_W(\lambda) - 2\cos(\pi/r)$ all lie within a certain small finite set.

In the case of interest here, $(p, q, r) = (2, 3, 2)$, X, Y have traces 0, 1 respectively, and we are interested in the roots of $\tau_W(\lambda)$ itself. It can moreover be readily shown that in this case $\tau_W(\lambda)$ is a monic polynomial with integer coefficients, and is odd or even depending on the parity of the length parameter ℓ .

In our case, ℓ is odd, so τ_W is odd. Moreover, the roots of τ_W that correspond to elementary representations are 0 (corresponding to a representation onto S_3), ± 1 (A_4), $\pm\sqrt{2}$ (S_4), $\pm\sqrt{3}$ (\mathbb{Z}_6 or a parabolic subgroup of the form $\mathbb{Z}^2 \rtimes \mathbb{Z}_6$) and $\frac{\pm 1 \pm \sqrt{5}}{2}$ (A_5).

Since τ_W is odd, the negative of any root is also a root of the same multiplicity. Since τ_W has integer coefficients, $\frac{1+\sqrt{5}}{2}$ is a root if and only if $\frac{1-\sqrt{5}}{2}$ is a root, in which case they have the same multiplicity. It follows therefore that G admits a non-abelian free subgroup except possibly in the cases where

$$(3) \quad \tau_W(\lambda) = \lambda^a(\lambda^2 - 1)^b(\lambda^2 - 2)^c(\lambda^2 - 3)^d(\lambda^4 - 3\lambda^2 + 1)^e$$

for some non-negative integers a, b, c, d, e with a odd.

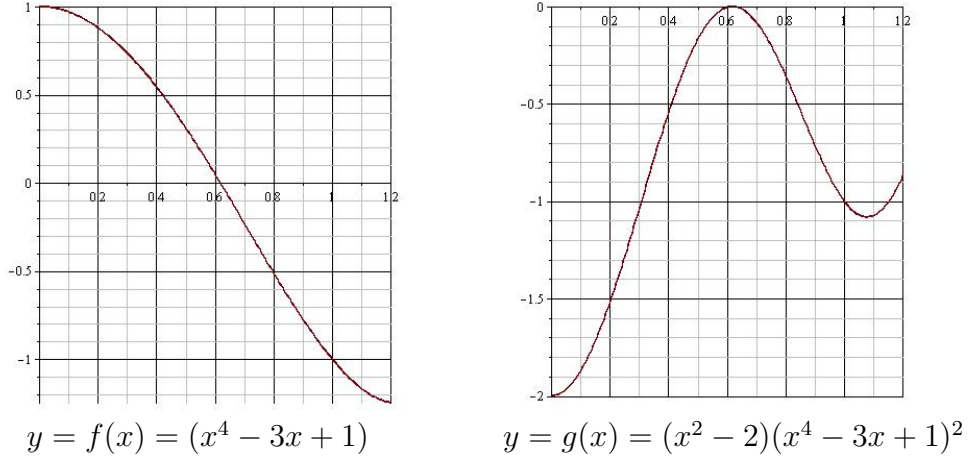
We next note one further property of the trace polynomial.

Lemma 3. *For any real number z with $|z| \leq \sqrt{3}$, $|\tau_W(z)| \leq 2$.*

Proof. The bounds on z mean that it is possible to find matrices $X, Y \in \mathrm{SU}(2)$ with $\mathrm{Tr}(X) = 0$, $\mathrm{Tr}(Y) = 1$ and $\mathrm{Tr}(XY) = z$. Hence $\tau_W(z) = \mathrm{Tr}(W(X, Y)) \in [-2, 2]$ since $W(X, Y) \in \mathrm{SU}(2)$. \square

Recall our assumption that the exponent-sum of y in W is not divisible by 3. It follows that there can be no essential representations from G onto \mathbb{Z}_6 or onto A_4 , and hence $b = d = 0$. Thus (3) becomes in this case

$$\tau_W(\lambda) = \lambda^a(\lambda^2 - 2)^c(\lambda^4 - 3\lambda^2 + 1)^e.$$

FIGURE 1. Graphs of f and g

In particular, $\tau_W(\sqrt{3}) = 3^{a/2}$, so we also have $a \leq 1$ by Lemma 3. Since a is odd this actually means $a = 1$. So the precise form of the trace polynomial in this case depends only on the two integers c, e . We find bounds on c, e as follows.

Theorem 4. *Let G be a generalised triangle group of type $(2, 3, 2)$ with odd length parameter ℓ such that the exponent-sum of y in $W(x, y)$ is not divisible by 3. Suppose that G does not contain a non-abelian free subgroup. Then*

$$\tau_W(\lambda) = \lambda(\lambda^2 - 2)^c(\lambda^4 - 3\lambda^2 + 1)^e$$

with

$$c \leq 4 \text{ and } e \leq 2c + 2.$$

In particular W has length parameter

$$\ell = 1 + 2c + 4e \leq 49.$$

Proof. The form of $\tau_W(\lambda)$ is as stated in the theorem, from the preceding remarks. We must check the inequalities relating c, e .

Let $f(\lambda) := (\lambda^4 - 3\lambda^2 + 1)$ and $g(\lambda) := (\lambda^2 - 2)f(\lambda)^2$. Let $\lambda_0 = 0.1$ and $\lambda_1 := 1.15$. Then calculations show that

$$|f(\lambda_1)| \sim 1.22 > 1 > 0.97 \sim |f(\lambda_0)|, \quad |g(\lambda_0)| \sim 1.87 > 1 \text{ and } |g(\lambda_1)| \sim 1.01 > 1.$$

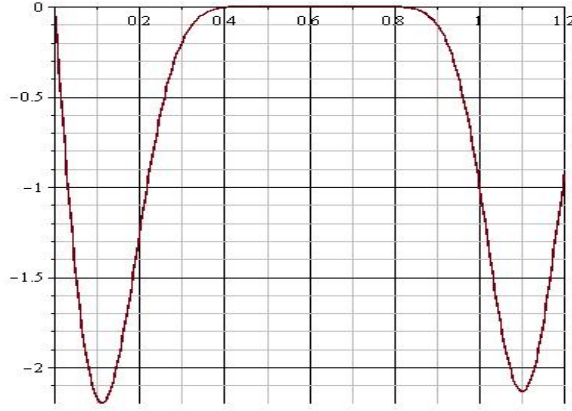
Now put

$$\sigma_0 := \lambda_0(\lambda_0^2 - 2)^5(\lambda_0^4 - 3\lambda_0^2 + 1)^{12}$$

and

$$\sigma_1 := \lambda_1(\lambda_1^4 - 3\lambda_1^2 + 1)^3.$$

Then further calculations show that $|\sigma_0| \sim 2.17 > 2$ and $|\sigma_1| \sim 2.08 > 2$.

FIGURE 2. Graph of $\sigma_0(x) = x(x^2 - 2)^5(x^4 - 3x + 1)^{12}$

Hence if τ_W has the above form with $e \geq 2c + 3$, we have

$$|\tau_W(\lambda_1)| = |\sigma_1| \times |g(\lambda_1)|^c \times |f(\lambda_1)|^{e-2c-3} > 2,$$

a contradiction to Lemma 3, while if $c \geq 5$ and $e \leq 2c + 2$ then

$$|\tau_W(\lambda_0)| = \frac{|\sigma_0| \times |g(\lambda_0)|^{c-5}}{|f(\lambda_0)|^{2c+2-e}} > 2,$$

also a contradiction to Lemma 3. □

4. SEARCHING

The bounds obtained on c and e (and hence also ℓ) in Theorem 4 show that only finitely many words have trace polynomial of the appropriate form. In principle it is possible to determine all such words by a computer search. In order to make such a search more practical, we first prove a result that restricts the form of words that can arise. We then describe a search algorithm that will identify all words with the correct form of trace polynomial, and finally give the results of this search.

4.1. Restricting the form of W . Up to cyclic permutation, we may write a word W in the form $(xy)^\ell$, $(xy^2)^\ell$ or

$$W = (xy)^{b(1)}(xy^2)^{b(2)} \dots (xy)^{b(t-1)}(xy^2)^{b(t)}$$

with $b(j) \geq 1$ for $j = 1, \dots, t$. Each maximal subword of the form $(xy)^{b(j)}$ or $(xy^2)^{b(j)}$ will be called a *block*, and the index $b(j)$ will be called the *length* of the block. Note that the number of blocks is either 1 or an even number.

Lemma 5. *If W has the form $(xy)^\ell$ or $(xy^2)^\ell$ and $\tau_W(\lambda)$ has the form*

$$\lambda(\lambda^2 - 2)^c(\lambda^4 - 3\lambda^2 + 1)^e$$

then $\ell = 1$ and $c = e = 0$. If $\tau_W(\lambda)$ has the form

$$\lambda(\lambda^2 - 2)^c(\lambda^4 - 3\lambda^2 + 1)^e$$

with $c + e > 0$ then W has $2e + 2$ blocks, of which $c + e$ have length greater than 1. In particular $e \geq c - 2$.

Proof. If $W = (xy)^\ell$ then $2\cos(\pi/2\ell)$ is a root of τ_W , so τ_W can have the given form only for $\ell \leq 2$. But also $\ell = \deg(\tau_W)$ is odd, so the only possibility is that $\ell = 1$ and $c = e = 0$ as claimed.

Now suppose that $c + e > 0$. We analyse the coefficients of $\lambda^{\ell-2}$ and $\lambda^{\ell-4}$ in $\tau_W(\lambda)$, using the formulae in [6]. Write

$$W = xy^{\alpha(1)} \dots xy^{\alpha(\ell)}$$

with $\alpha(j) \in \{1, 2\}$ for each j . Define $\beta(j) = -\exp(i\pi(\alpha(j+1) - \alpha(j))/3)$ (where j is interpreted modulo ℓ). Then the coefficient B_1 of $\lambda^{\ell-2}$ is the sum of all the $\beta(j)$: $B_1 = \sum_j \beta(j)$, while the coefficient B_2 of $\lambda^{\ell-4}$ is the sum of products $\beta(j)\beta(k)$, taken over all unordered pairs $\{j, k\}$ with $k \notin \{j-1, j, j+1\} \pmod{\ell}$.

As noted in [6], these coefficients satisfy the following equation:

$$B_1^2 = 2B_2 + \sum_j \beta(j)^2 + 2 \sum_j \beta(j)\beta(j+1).$$

We can also compute the coefficients directly from the formula

$$\tau_W(\lambda) = \lambda(\lambda^2 - 2)^c(\lambda^4 - 3\lambda^2 + 1)^e,$$

giving

$$B_1 = -2c - 3e; \quad B_2 = 4 \frac{c(c-1)}{2} + 9 \frac{e(e-1)}{2} + 6ce + e.$$

Putting these equations together gives

$$\sum_j \beta(j)^2 + 2 \sum_j \beta(j)\beta(j+1) = 4c + 7e.$$

Now the equation $B_1 = -2c - 3e$ means that, of the $\ell = 1 + 2c + 4e$ values of j , precisely $2c + 2e - 1$ give $\beta(j) = -1$, while $e + 1$ give $\beta(j) = -\exp(i\pi/3)$ (corresponding to points in W where xy is followed by xy^2) and $e + 1$ give $\beta(j) = -\exp(-i\pi/3)$ (places where xy^2 is followed by xy). It then follows that W can (up to cyclic conjugacy) be subdivided into $2e + 2$ blocks, which are alternatingly powers of xy and of xy^2 .

It also follows that

$$\sum_j \beta(j)^2 = (2c + 2e - 1) - (e + 1) = 2c + e - 2,$$

and hence

$$\sum_j \beta(j)\beta(j+1) = c + 3e + 1.$$

Now consider the above subdivision of W into blocks. If a block is a power $(xy)^s$ or $(xy^2)^s$ with $s > 1$, then the start of the block reads xy^2xyxy or $xyxy^2xy^2$, giving a value of $\beta(j)\beta(j+1)$ of $\exp(-i\pi/3)$ or $\exp(i\pi/3)$ respectively, while the end of the block reads $xyxyxy^2$ or xy^2xy^2xy , giving $\beta(j)\beta(j+1) = \exp(i\pi/3)$ or $\exp(-i\pi/3)$ respectively. Thus each block $(xy)^s$ or $(xy^2)^s$ with $s > 1$ gives rise to one value $\beta(j)\beta(j+1) = \exp(i\pi/3)$ and one value $\beta(j)\beta(j+1) = \exp(-i\pi/3)$ – totalling 1. All other values of $\beta(j)\beta(j+1)$ are precisely 1, so $k - \sum_j \beta(j)\beta(j+1) = c + e$ is equal to the number of blocks $(xy)^s$ or $(xy^2)^s$ for which $s > 1$.

Finally, since the total number of blocks is $2e+2$, it follows that $c+e \leq 2e+2$, or $c-2 \leq e$, as required. \square

4.2. The search algorithm. Suppose given non-negative integers c, e with $c+e > 0$. We wish to find all words W in $\mathbb{Z}_2 * \mathbb{Z}_3 = \langle x, y | x^2 = y^3 = 1 \rangle$ whose trace polynomial is

$$\tau_W(\lambda) = \lambda(\lambda^2 - 2)^c(\lambda^4 - 3\lambda^2 + 1)^e.$$

By Lemma 5 it suffices to consider words of the form

$$W = (xy)^{b(1)}(xy^2)^{b(2)} \dots (xy)^{b(t-1)}(xy^2)^{b(t)}$$

with $t = 2e + 2$ such that $b(j) > 1$ for precisely $c + e$ values of j – or equivalently $b(j) = 1$ for precisely $e + 2 - c$ values of j . We encode W as a list of t positive integers $[b(1), \dots, b(t)]$, of which $e + 2 - c$ are equal to 1. Since we are interested in classifying W only up to cyclic permutation and inversion, it suffices also to consider lists only up to cyclic permutation and reversal.

In practice, we construct our list using GAP [2] from two pieces of input: a list L of $c + e$ positive integers totalling $2c + 2e - 1$; and a subset C of $\{1, \dots, 2e + 2\}$ of size $e + 2 - c$. The final list is obtained by adding 1 to each element of L and inserting a 1 in the positions C . This process has two advantages: the sets of possible inputs \mathcal{L}, \mathcal{C} for L, C respectively are of more manageable size than the overall input set $\mathcal{L} \times \mathcal{C}$; and the resulting double programming loop facilitates parallelisation of the process, making implementation possible in reasonable time.

Having constructed our list of integers, we use it to calculate $\tau_W(\lambda)$ and record only those lists that give the correct result. As a final step, we sift through the (now manageably small) list of recorded lists and delete repetitions up to cyclic permutation and reversal.

For smaller values of c and e , this algorithm can be carried out in a reasonable time-frame with the standard version of GAP. However, for higher values it was essential to use the parallel version of GAP. The longest search, with $(c, e) = (4, 10)$, needed to test all possible lists (up to cyclic permutation and reversal) of 22 positive integers, such that precisely 8 entries are equal to 1, and whose sum is 49. In practice, 722030 lists were tested by 32 slave

processors, each working for around 31240 minutes (or just over 3 weeks), to confirm that no word has trace polynomial $\lambda(\lambda^2 - 2)^4(\lambda^4 - 3\lambda^2 + 1)^{10}$.

4.3. Search Results. The table below shows all words (up to equivalence) with trace polynomial of the form $\lambda(\lambda^2 - 2)^c(\lambda^4 - 3\lambda^2 + 1)^e$.

n	c	e	W_n
1	0	0	xy
2	0	1	$(xy)^2xy^2xyxy^2$
3	0	2	$(xy)^3xy^2xy(xy^2)^2xyxy^2$
4	1	0	$(xy)^2xy^2$
5	1	1	$(xy)^3xy^2xy(xy^2)^2$
6	1	2	$(xy)^3(xy^2)^3xyxy^2(xy)^2xy^2$
7	1	2	$(xy)^4(xy^2)^2xyxy^2xy(xy^2)^2$
8	1	3	$(xy)^4xy^2xy(xy^2)^3(xy)^2xy^2xy(xy^2)^2$
9	1	4	$(xy)^4xy^2xy(xy^2)^3xy(xy^2)^2xyxy^2(xy)^2(xy^2)^3$
10	1	4	$(xy)^4xy^2xy(xy^2)^3(xy)^3(xy^2)^2xy(xy^2)^2xyxy^2$
11	2	0	$(xy)^3(xy^2)^2$
12	2	1	$(xy)^3(xy^2)^3xy(xy^2)^2$
13	2	2	$(xy)^4(xy^2)^3(xy)^2(xy^2)^2xyxy^2$
14	2	3	$(xy)^4xy^2xy(xy^2)^2(xy)^3(xy^2)^3xy(xy^2)^2$
15	2	3	$(xy)^4(xy^2)^3xyxy^2(xy)^2(xy^2)^3(xy)^2xy^2$
16	2	3	$(xy)^4(xy^2)^3xy(xy^2)^3(xy)^2xy^2xy(xy^2)^2$
17	2	4	$(xy)^4(xy^2)^2xyxy^2xy(xy^2)^2(xy)^3(xy^2)^4(xy)^2xy^2$
18	2	4	$(xy)^4(xy^2)^3xy(xy^2)^2(xy)^3(xy^2)^3xyxy^2(xy)^2xy^2$
19	2	4	$(xy)^4(xy^2)^4xyxy^2xy(xy^2)^2xy(xy^2)^2(xy)^3(xy^2)^2$
20	3	1	$(xy)^4(xy^2)^3(xy)^2(xy^2)^2$
21	3	2	$(xy)^4(xy^2)^3xy(xy^2)^2(xy)^3(xy^2)^2$
22	3	2	$(xy)^4(xy^2)^3(xy)^2(xy^2)^3xy(xy^2)^2$
23	3	4	$(xy)^4xy^2xy(xy^2)^4(xy)^2(xy^2)^3(xy)^2xy^2(xy)^2(xy^2)^3$
24	3	6	$(xy)^4(xy^2)^4xyxy^2(xy)^2(xy^2)^3(xy)^3(xy^2)^2xy(xy^2)^3(xy)^3xy^2(xy)^2xy^2$
25	3	6	$(xy)^5(xy^2)^3xy(xy^2)^2xyxy^2(xy)^3(xy^2)^4(xy)^2xy^2xy(xy^2)^2(xy)^3(xy^2)^2$
26	3	8	$(xy)^5(xy^2)^4(xy)^3(xy^2)^2xyxy^2(xy)^2xy^2(xy)^3(xy^2)^3(xy)^2xy^2xy(xy^2)^4 \cdot$ $(xy)^2(xy^2)^2xyxy^2$
27	4	4	$(xy)^4(xy^2)^3(xy)^2xy^2(xy)^3(xy^2)^2(xy)^3(xy^2)^4xy(xy^2)^2$
28	4	4	$(xy)^4(xy^2)^3xy(xy^2)^2(xy)^3(xy^2)^4(xy)^2(xy^2)^3(xy)^2xy^2$
29	4	4	$(xy)^4(xy^2)^4(xy)^2(xy^2)^3xy(xy^2)^2(xy)^3(xy^2)^3(xy)^2xy^2$
30	4	5	$(xy)^4(xy^2)^3(xy)^2(xy^2)^2xy(xy^2)^4(xy)^4(xy^2)^2(xy)^3(xy^2)^2xyxy^2$
31	4	6	$(xy)^4(xy^2)^2xyxy^2(xy)^2xy^2(xy)^3(xy^2)^4xy(xy^2)^3(xy)^4(xy^2)^3(xy)^2(xy^2)^2$

Table 1

List of words with trace polynomial $\lambda(\lambda^2 - 2)^c(\lambda^4 - 3\lambda^2 + 1)^e$.

5. SMALL CANCELLATION

Several of the groups in Table 1 satisfy a small cancellation condition. Recall that a word U is a *piece* of a word W (in the free product $\mathbb{Z}_2 * \mathbb{Z}_3$) if there are distinct words V_1, V_2 such that each of $U \cdot V_1$ and $U \cdot V_2$ is cyclically reduced as written and a cyclic permutation of $W^{\pm 1}$. The table below expresses the words concerned (up to cyclic permutation) as a product of 3 non-pieces, each of even length at least 8. Following [5, Corollary 2.3] this is enough to confirm the existence of non-abelian free subgroups.

$n.$	W_n
9	$[(xy)^4] \cdot [xy^2xy(xy^2)^3xy] \cdot [(xy^2)^2xyxy^2(xy)^2(xy^2)^3]$
10	$[(xy)^4] \cdot [xy^2xy(xy^2)^3xy] \cdot [(xy)^2(xy^2)^2xy(xy^2)^2xyxy^2]$
14	$[(xy)^4] \cdot [xy^2xy(xy^2)^2(xy)^3xy^2] \cdot [(xy^2)^2xy(xy^2)^2]$
15	$[(xy)^4] \cdot [(xy^2)^3xyxy^2xy] \cdot [xy(xy^2)^3(xy)^2xy^2]$
17	$[(xy)^2(xy^2)^2xy] \cdot [xy^2xy(xy^2)^2(xy)^3xy^2] \cdot [(xy^2)^3(xy)^2xy^2(xy)^2]$
18	$[(xy)^4] \cdot [(xy^2)^3xy(xy^2)^2(xy)^2] \cdot [xy(xy^2)^3xyxy^2(xy)^2xy^2]$
19	$[(xy)^4(xy^2)^4xyxy^2] \cdot [xy(xy^2)^2xyxy^2] \cdot [xy^2(xy)^3(xy^2)^2]$
22	$[(xy)^4] \cdot [(xy^2)^3(xy)^2(xy^2)^2] \cdot [xy^2xy(xy^2)^2]$
23	$[xyxy^2xy(xy^2)^4(xy)^2xy^2] \cdot [(xy^2)^2(xy)^2xy^2xy] \cdot [xy(xy^2)^3(xy)^3]$
24	$[(xy)^3(xy^2)^4xyxy^2xy] \cdot [xy(xy^2)^3(xy)^3(xy^2)^2xy(xy^2)^2] \cdot [xy^2(xy)^3xy^2(xy)^2xy^2xy]$
25	$[(xy)^5] \cdot [(xy^2)^3xy(xy^2)^2] \cdot [xyxy^2(xy)^3(xy^2)^4(xy)^2xy^2xy(xy^2)^2(xy)^3(xy^2)^2]$
26	$[(xy)^5] \cdot [(xy^2)^4(xy)^3] \cdot [(xy^2)^2xyxy^2(xy)^2xy^2(xy)^3(xy^2)^3(xy)^2xy^2xy(xy^2)^4(xy)^2(xy^2)^2xyxy^2]$
27	$[(xy)^4(xy^2)^3(xy)^2xy^2xy] \cdot [(xy)^2(xy^2)^2xy] \cdot [(xy)^2(xy^2)^4xy(xy^2)^2]$
28	$[(xy^2)^3xy(xy^2)^2(xy)^3(xy^2)^3] \cdot [xy^2(xy)^2(xy^2)^2] \cdot [xy^2(xy)^2xy^2(xy)^4]$
30	$[(xy)^4(xy^2)^3] \cdot [(xy)^2(xy^2)^2xy(xy^2)^2] \cdot [(xy^2)^2(xy)^4(xy^2)^2(xy)^3(xy^2)^2xyxy^2]$
31	$[(xy)^4(xy^2)^2xy] \cdot [xy^2(xy)^2xy^2xy] \cdot [(xy)^2(xy^2)^4xy(xy^2)^3(xy)^4(xy^2)^3(xy)^2(xy^2)^2]$

Table 2
Words from Table 1 satisfying the small cancellation condition.

6. ALMOST SMALL CANCELLATION

The group $G = G_{13}$ in our table does not satisfy a small cancellation condition, but comes sufficiently close for small-cancellation methods to apply.

Thus while G does not satisfy the small cancellation condition $C(6)$ (as a quotient of the free product $\mathbb{Z}_2 * \mathbb{Z}_3$), it does satisfy $C(5)$ in a fairly strong form. We will exploit this to obtain the existence of a non-abelian free subgroup.

Theorem 6. *Let $G = \langle x, y | x^2 = y^3 = ((xy)^4(xy^2)^3(xy)^2(xy^2)^2xyxy^2)^2 = 1 \rangle$. Let $N \gg 1$ be an integer. Then $A := (xy)^N$ and $B := (xy^{-1})^N$ freely generate a free subgroup of G .*

As the proof of this result is substantial and uses technology which is different from the main thrust of the paper, we have included it in Appendix C below.

7. AD-HOC ARGUMENTS

By §5 and Theorem 6 we have 14 exceptional words to consider, as follows.

n	W_n
1	xy
2	$(xy)^2xy^2xyxy^2$
3	$(xy)^3xy^2xy(xy^2)^2xyxy^2$
4	$(xy)^2xy^2$
5	$(xy)^3xy^2xy(xy^2)^2$
6	$(xy)^3(xy^2)^3xyxy^2(xy)^2xy^2$
7	$(xy)^4(xy^2)^2xyxy^2xy(xy^2)^2$
8	$(xy)^4xy^2xy(xy^2)^3(xy)^2xy^2xy(xy^2)^2$
11	$(xy)^3(xy^2)^2$
12	$(xy)^3(xy^2)^3xy(xy^2)^2$
16	$(xy)^4(xy^2)^3xy(xy^2)^3(xy)^2xy^2xy(xy^2)^2$
20	$(xy)^4(xy^2)^3(xy)^2(xy^2)^2$
21	$(xy)^4(xy^2)^3xy(xy^2)^2(xy)^3(xy^2)^2$
29	$(xy)^4(xy^2)^4(xy)^2(xy^2)^3xy(xy^2)^2(xy)^3(xy^2)^3(xy)^2xy^2$

Table 3
Exceptional words.

Let us denote by G_n the group

$$\langle x, y | x^2 = y^3 = W_n(x, y)^2 = 1 \rangle,$$

where W_n is the word listed in Table 1. The Rosenberger Conjecture is known for words with length parameter $\ell \leq 6$ [9]. This covers groups G_1, G_2, G_4 and G_{11} . Indeed, G_1, G_2, G_4 are finite of orders 6, 720 and 48 respectively, while G_{11} is an extension of \mathbb{Z}^3 by A_4 – see Appendix B below for a verification of this using GAP.

Groups G_3, G_7 and G_{29} can be shown to be large using GAP [2] – see Appendix B. (Recall that a group G is *large* if some subgroup $H < G$ of finite index admits an epimorphism $H \twoheadrightarrow F_2$ onto the free group F_2 of rank 2. Large groups clearly contain non-abelian free subgroups.) Now the relators of G_6 are consequences of those of G_3 , so it follows that group G_3 is a homomorphic image of G_6 . Since G_3 is large, G_6 is also large.

Lemma 7. *Group G_5 contains a non-abelian free subgroup.*

Proof. Group G_5 was shown to be infinite by Lévai, Rosenberger and Souvignier [7]. Their proof constructs a representation $\rho : G' \rightarrow SL(3, \mathbb{C})$ of the commutator subgroup G' , the image of which is generated by two matrices X, Y of order 3. A more detailed analysis of the matrices in [7] shows that XY (and hence also YX) has eigenvalues $+1, -1, -1$, with the (-1) -eigenspace having dimension 1. Specifically,

$$XY = \begin{pmatrix} -2 & -1 & t^{-1} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where t is an algebraic number with $t^6 - 3t^3 + 1 = 0$. Let V be the plane spanned by the eigenvectors of XY , namely $v_{-1} := (1, -1, 0)^T$ and $v_{+1} := (1, 1, 4t)^T$. Then a calculation shows that $YX(v_{-1}) \notin V$ so V is not invariant under YX . If V' is the plane spanned by the eigenvectors of YX , then $V \neq V'$ so $L := V \cap V'$ is a line. Now $(XY)^2$ and $(YX)^2$ fix V and V' respectively (pointwise), and so they both fix L . Under the induced action on the quotient plane \mathbb{C}^3/L , $(XY)^2$ and $(YX)^2$ are parabolic with distinct fixed subspaces. Hence they generate a non-elementary subgroup of $PSL(2, \mathbb{C})$. This subgroup, and hence also G , contains a non-abelian free subgroup. \square

Corollary 8. *Each of the groups G_8, G_{12}, G_{16} and G_{21} contains a non-abelian free subgroup.*

Proof. It is an exercise to show that, for $j = 8, 12, 16, 21$, the element of G_5 represented by W_j^2 is trivial. Hence G_5 is a homomorphic image of G_j . Since G_5 contains a non-abelian free subgroup, so does G_j . \square

8. CONCLUSION

Putting together the results of §§4, 5 and 7 we obtain

Theorem 9. *Let*

$$G = \langle x, y | x^2 = y^3 = W(x, y)^2 = 1 \rangle$$

be a generalised triangle group, where the exponent-sums of x, y in W are coprime to 2, 3 respectively. Then either G contains a non-abelian free subgroup, or G has a soluble subgroup of finite index, except possibly where G is isomorphic to the group

$$G_{20} = \langle x, y | x^2 = y^3 = ((xy)^4(xy^2)^3(xy)^2(xy^2)^2)^2 = 1 \rangle.$$

This, together with the observations on the even-length case in §2 and in Appendix A, complete the proof of Theorem 1.

APPENDIX A: EVEN LENGTH RELATORS

In [4] the Rosenberger conjecture was verified for generalised triangle groups of the form

$$G = \langle x, y | x^2 = y^2 = W(x, y)^2 = 1 \rangle$$

in which x appears in W with even exponent-sum, with precisely six exceptions. The exceptional groups are defined by setting W to be one of

- (1) $W_{7a} := (xy)^4(xy^2)^3(xy)^2xy^2$;
- (2) $W_{7b} := (xy)^3(xy^2)^2xy(xy^2)^2xyxy^2$;
- (3) $W_{9a} := (xy)^5(xy^2)^3(xy)^2xy^2xy(xy^2)^2$;
- (4) $W_{12} := (xy)^4(xy^2)^2xy(xy^2)^3(xy)^2xy^2xy(xy^2)^2$;
- (5) $W_{13a} := (xy)^4(xy^2)^4xy(xy^2)^3(xy)^2xy^2xy(xy^2)^2$;
- (6) $W_{15a} := (xy)^4(xy^2)^4xy(xy^2)^2xy(xy^2)^3(xy)^3(xy^2)^2xyxy^2$.

Here we give the computational details of proofs that all of the above, with the possible exception of W_{7a} , give rise to groups containing non-abelian free subgroups.

Jack Button (private communication) informed us that he could prove, using his largeness-testing software [1], that the groups with $W \in \{W_{9a}, W_{12}, W_{13a}, W_{15a}\}$ are large – and hence in particular contain non-abelian free subgroups. Below is a log of a GAP session verifying this fact.

```
gap> Epi:=function(P,ll)
> local i,gg,Q;
> Q:=ShallowCopy(P);
> gg:=GeneratorsOfPresentation(Q);
> for i in ll do if i<1+Length(gg) then AddRelator(Q,gg[i]); fi; od;
> TzGoGo(Q);
> TzPrint(Q);
> return(Q);
> end;
function( P, ll ) ... end
gap> F:=FreeGroup(2);; x:=F.1;; y:=F.2;; U:=x*y;; V:=x*y^2;;
gap> W9:=U^5*V^3*U^2*V*U*V^2;;
gap> W12:=U^4*V^2*U*V^3*U^2*V*U*V^2;;
gap> W13:=U^4*V^4*U*V^3*U^2*V*U*V^2;;
gap> W15:=U^4*V^4*U*V^2*U*V^3*U^3*V^2*U*V;;
gap> G9:=F/[x^2,y^3,W9^2];;
gap> G12:=F/[x^2,y^3,W12^2];;
gap> G13:=F/[x^2,y^3,W13^2];;
gap> G15:=F/[x^2,y^3,W15^2];;
gap> x:=G9.1;; y:=G9.2;;
gap> H9:=Subgroup(G9,[x,(y*x)^2*(y*x^-1)^2*y,y*x*y^-1*x*(y*x^-1)^2*y^-1,
(y*x)^2*y^-1*x*y^-1*x^-1*y*x^-1*y^-1*x^-1*y]);;
gap> x:=G12.1;; y:=G12.2;;
```

```

gap> H12:=Subgroup(G12,[x,(y*x)^2*(y*x^-1)^2*y,y*x*y^-1*x*(y*x^-1)^2*y^-1,
  (y*x)^2*y^-1*x*y^-1*x^-1*y*x^-1*y^-1*x^-1*y ]);;
gap> P9:=PresentationSubgroup(G9,H9);
<presentation with 6 gens and 14 rels of total length 134>
gap> P12:=PresentationSubgroup(G12,H12);
<presentation with 6 gens and 14 rels of total length 154>
gap> Index(G9,H9);
20
gap> Index(G12,H12);
20
gap> Epi(P9,[4,6]);
#I  there are 2 generators and 1 relator of total length 2
#I  generators: [ _x1, _x2 ]
#I  relators:
#I  1.  2  [ 1, 1 ]
<presentation with 2 gens and 1 rels of total length 2>
gap> Epi(P12,[4,6]);
#I  there are 2 generators and 1 relator of total length 2
#I  generators: [ _x1, _x2 ]
#I  relators:
#I  1.  2  [ 1, 1 ]
<presentation with 2 gens and 1 rels of total length 2>
gap> H13:=Subgroup(G13,[(G13.1*G13.2)^5]);;
gap> P13:=PresentationNormalClosure(G13,H13);
<presentation with 11 gens and 30 rels of total length 396>
> gap> Epi(P13,[1..4]);
gap> Epi(P13,[1..4]);
#I  there are 2 generators and 0 relators of total length 0
#I  generators: [ _x5, _x6 ]
#I  there are no relators
<presentation with 2 gens and 0 rels of total length 0>
gap> H15:=Subgroup(G15,[(G15.1*G15.2)^5]);;
gap> P15:=PresentationNormalClosure(G15,H15);
<presentation with 11 gens and 30 rels of total length 484>
gap> Epi(P15,[1..4]);
#I  there are 2 generators and 1 relator of total length 8
#I  generators: [ _x5, _x8 ]
#I  relators:
#I  1.  8  [ -2, 1, 2, 2, 1, -2, -1, -1 ]
<presentation with 2 gens and 1 rels of total length 8>
gap> #
gap> # Each group G9, G12, G13, G15 has a finite index subgroup that admits

```

```
gap> # an epimorphism onto a large group.
gap> # Therefore each is itself large.
gap> #
```

In the case of the group defined by $W = W_{7b}$, while we are unable to prove largeness, we can verify the existence of non-abelian free subgroups (and hence the Rosenberger conjecture) by constructing an irreducible essential representation σ to $PSL(2, \mathbb{C})$ from the commutator subgroup $[G, G]$ of G (which has index 6). This is outlined in the proof of Theorem 2. The computational details are as follows. First we use GAP [2] to obtain the presentation of $[G, G]$ stated in the proof of Theorem 2:

```
gap> #
gap> # Calculation of presentation of derived subgroup [G,G] of
gap> #
gap> # G = < x, y | x^2 = y^3 = W^2 = 1 >, where
gap> #
gap> # W = (xy)^3 (xy^2)^2 xy (xy^2)^2 xy xy^2
gap> #
gap> F:=FreeGroup(["x","y"]);; x:=F.1;; y:=F.2;;
gap> W:=(x*y)^3*(x*y^2)^2*x*y*(x*y^2)^2*x*y*x*y^2;;
gap> G:=F/[x^2,y^3,W^2];;
gap> D:=DerivedSubgroup(G);;
gap> P:=PresentationSubgroup(G,D);;
gap> TzPrint(P);
#I generators: [ _x1, _x2 ]
#I relators:
#I 1. 12 [ 2, 2, 1, -2, 1, 1, 2, 2, 1, -2, 1, 1 ]
#I 2. 12 [ 2, 2, 1, 1, -2, 1, 2, 2, 1, 1, -2, 1 ]
#I 3. 14 [ 2, 2, -1, -1, -1, 2, -1, 2, 2, -1, -1, -1, 2, -1 ]
#I 4. 14 [ 2, 2, -1, 2, -1, -1, -1, 2, 2, -1, 2, -1, -1, -1 ]
#I 5. 14 [ 2, 2, -1, 2, -1, 2, 1, 2, 2, -1, 2, -1, 2, 1 ]
#I 6. 14 [ 2, 2, 1, 2, -1, 2, -1, 2, 2, 1, 2, -1, 2, -1 ]
gap> #
```

This confirms that $[G, G]$ is presented on two generators u, v by six defining relators of the form W_j^2 , $j = 1, \dots, 6$. To check that the two matrices

$$M = \begin{pmatrix} 0 & -1 \\ 1 & a \end{pmatrix}, \quad N = \begin{pmatrix} -z & 1 + bz + z^2 \\ -1 & z + b \end{pmatrix}$$

in the proof of Theorem 2 do indeed give a representation σ to $PSL(2, \mathbb{C})$, we need to ensure that $W_j(M, N)$ has trace 0 in $\mathbb{Q}[z]$ for each j . In the MAPLE session illustrated below we verify that $W_1(M, N)$ and $W_2(M, N)$ have the equal trace (as a polynomial in a, b, z) which we define to be $f(a, b, z)$. Similarly $W_3(M, N)$ and $W_4(M, N)$ have equal trace $g(a, b, z)$, while $W_5(M, N)$ and $W_6(M, N)$ have equal trace $h(a, b, z)$.

with(*LinearAlgebra*) :

$$M := \begin{bmatrix} 0 & -1 \\ 1 & a \end{bmatrix}; N := \begin{bmatrix} -z & bz + z^2 + 1 \\ -1 & z + b \end{bmatrix}; m := M^{-1}; n := N^{-1};$$

$$\begin{bmatrix} a & 1 \\ -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} z + b & -bz - z^2 - 1 \\ 1 & -z \end{bmatrix} \tag{1}$$

$$W := N.N.M.n.M.M : f := W[1, 1] + W[2, 2] : W := N.N.M.M.n.M : \text{simplify}(W[1, 1] + W[2, 2] - f);$$

$$0 \tag{2}$$

$$W := N.N.m.m.m.N.m : g := W[1, 1] + W[2, 2] : W := N.N.m.N.m.m.m : \text{simplify}(W[1, 1] + W[2, 2] - g);$$

$$0 \tag{3}$$

$$W := N.N.m.N.m.N.M : h := W[1, 1] + W[2, 2] : W := N.N.M.N.m.N.m : \text{simplify}(W[1, 1] + W[2, 2] - h);$$

$$0 \tag{4}$$

We next set

$$p(X) := X^{12} - 3X^{10} - 9X^8 + 42X^6 - 48X^4 + 15X^2 + 1,$$

and compute a Gröbner basis B for the ideal J of the polynomial ring $R := \mathbb{Q}[a, b, z]$ generated by $f, g, h, p(a)$. The first term in B is a polynomial $q(z)$ of degree 48, which we confirm to be irreducible. The second term is a polynomial in b, z which is linear in b , and the third term is linear in a . Hence the quotient ring $K := R/J$ is an algebraic extension of \mathbb{Q} of degree 48, generated by z . This establishes that $\sigma : u \mapsto M, v \mapsto N$ defines a representation from $[G, G]$ to $PSL(2, K)$.

Using a second Groebner basis calculation, we show that the ideal of R generated by B together with $d := a^2 + b^2 + c^2 + abc - 4$ (where c is the trace of MN) is the whole of R . This confirms that σ is an irreducible representation.

Finally, we check that p is irreducible, and that both $a, b \in K$ are roots of p . It follows that none of the eigenvalues of M, N can be an n 'th root of unity for any n with Euler totient $\phi(n) < 2 \cdot \text{degree}(p) = 24$, and hence that each of M, N has order greater than 12 in $PSL(2, K)$. In particular, $\sigma([G, G]) = \langle M, N \rangle$ cannot be dihedral or isomorphic to one of the finite subgroups A_4, S_4, A_5 of $PSL(2, \mathbb{C})$.

This is sufficient to confirm that $\sigma([G, G])$ is non-elementary and so contains a non-abelian free subgroup.

with(Groebner) :
 $p := a^{12} - 3 \cdot a^{10} - 9 \cdot a^8 + 42 \cdot a^6 - 48 \cdot a^4 + 15 \cdot a^2 + 1 :$
 $B := \text{Basis}([f, g, h, p], \text{plex}(a, b, z)) : q := B[1] ;$

$$z^{48} + 36 z^{46} + 612 z^{44} + 6664 z^{42} + 52650 z^{40} + 328836 z^{38} + 1699012 z^{36} + 7462230 z^{34} + 28434543 z^{32} + 95050056 z^{30} + 281191314 z^{28} + 738204180 z^{26} + 1711054260 z^{24} + 3474132456 z^{22} + 6070609875 z^{20} + 8848833456 z^{18} + 10422584295 z^{16} + 9692682042 z^{14} + 7001073856 z^{12} + 3861624912 z^{10} + 1588370235 z^8 + 469516630 z^6 + 92723331 z^4 + 10731186 z^2 + 491401$$
 (1)

$\text{degree}(B[2], a); \text{degree}(B[2], b); \text{degree}(B[3], a);$

$$\begin{array}{c} 0 \\ 1 \\ 1 \end{array}$$
 (2)

$\text{factors}(p);$

$$[1, [[a^{12} - 3 a^{10} - 9 a^8 + 42 a^6 - 48 a^4 + 15 a^2 + 1, 1]]]$$
 (3)

$\text{factors}(q);$

$$[1, [[z^{48} + 36 z^{46} + 612 z^{44} + 6664 z^{42} + 52650 z^{40} + 328836 z^{38} + 1699012 z^{36} + 7462230 z^{34} + 28434543 z^{32} + 95050056 z^{30} + 281191314 z^{28} + 738204180 z^{26} + 1711054260 z^{24} + 3474132456 z^{22} + 6070609875 z^{20} + 8848833456 z^{18} + 10422584295 z^{16} + 9692682042 z^{14} + 7001073856 z^{12} + 3861624912 z^{10} + 1588370235 z^8 + 469516630 z^6 + 92723331 z^4 + 10731186 z^2 + 491401, 1]]]$$
 (4)

$W := M.N : c := W[1, 1] + W[2, 2] : d := a^2 + b^2 + c^2 + a \cdot b \cdot c - 4 :$
 $\text{Basis}([f, g, h, p, d], \text{plex}(a, b, z));$

$$[1]$$
 (5)

with(PolynomialIdeals) :
 $J := \langle B \rangle : \text{EliminationIdeal}(J, \{a\}); \text{EliminationIdeal}(J, \{b\});$

$$\langle a^{12} - 3 a^{10} - 9 a^8 + 42 a^6 - 48 a^4 + 15 a^2 + 1 \rangle$$

$$\langle b^{12} - 3 b^{10} - 9 b^8 + 42 b^6 - 48 b^4 + 15 b^2 + 1 \rangle$$
 (6)

APPENDIX B: GAP CALCULATIONS IN THE ODD LENGTH CASE

Verification that G_{11} is abelian-by-finite.

```
gap> F:=FreeGroup(["x","y"]);; x:=F.1;; y:=F.2;;
gap> W:=(x*y)^2*(x*y^2)^3;;
gap> G:=F/[x^2,y^3,W^2];;
gap> H:=Subgroup(G,[(G.1*G.2)^4]);;
gap> P:=PresentationNormalClosure(G,H);
<presentation with 4 gens and 11 rels of total length 48>
gap> SimplifyPresentation(P);
#I there are 3 generators and 3 relators of total length 14
#I there are 3 generators and 3 relators of total length 12
gap> TzPrint(P);
#I generators: [ _x1, _x2, _x3 ]
#I relators:
#I 1. 4 [ 1, 2, -1, -2 ]
#I 2. 4 [ 1, 3, -1, -3 ]
#I 3. 4 [ 2, 3, -2, -3 ]
```

The kernel of a representation $G_{11} \rightarrow S_4$ is free abelian of rank 3.

Verification that G_3 is large.

```
gap> F:=FreeGroup(["x","y"]);; x:=F.1;; y:=F.2;;
gap> W:=(x*y)^3*x*y^2*x*y*(x*y^2)^2*x*y*x*y^2;;
gap> G:=F/[x^2,y^3,W^2];;
gap> H:=Subgroup(G,[(G.1*G.2)^5]);;
gap> P:=PresentationNormalClosure(G,H);
<presentation with 10 gens and 29 rels of total length 196>
gap> SimplifyPresentation(P);
#I there are 7 generators and 24 relators of total length 236
#I there are 7 generators and 24 relators of total length 214
gap> gg:=GeneratorsOfPresentation(P);;
gap> AddRelator(P,gg[7]);
gap> AddRelator(P,gg[4]*gg[6]^-1);
gap> for j in [1,2,3,5] do AddRelator(P,gg[1]*gg[j]); od;
gap> SimplifyPresentation(P);
#I there are 2 generators and 1 relator of total length 2
gap> TzPrint(P);
#I generators: [ _x1, _x5 ]
#I relators:
#I 1. 2 [ 1, 1 ]
```

The kernel of a representation $G_3 \rightarrow A_5$ has the large group $\mathbb{Z}_2 * \mathbb{Z}$ as a homomorphic image.

Verification that G_7 is large.

```
gap> F:=FreeGroup(["a","b"]);; a:=F.1;; b:=F.2;;
gap> W:=(a*b)^4*(a*b^2)^2*a*b*a*b^2*a*b*(a*b^2)^2;;
gap> G:=F/[a^2,b^3,W^2];;
gap> H:=Subgroup(G,[(G.1*G.2)^5]);;
gap> Q:=PresentationNormalClosure(G,H);
<presentation with 11 gens and 30 rels of total length 242>
gap> SimplifyPresentation(Q);
#I there are 7 generators and 26 relators of total length 438
#I there are 7 generators and 26 relators of total length 432
gap> gg:=GeneratorsOfPresentation(Q);
[ _x1, _x3, _x4, _x6, _x8, _x10, _x11 ]
gap> AddRelator(Q,gg[3]*gg[6]);
gap> for j in [1,2,5,7] do AddRelator(Q,gg[j]); od;
gap> SimplifyPresentation(Q);
gap> K:=FpGroupPresentation(Q);;
gap> N:=Subgroup(K,[K.1,K.2^2]);;
gap> P:=PresentationNormalClosure(K,N);
<presentation with 3 gens and 2 rels of total length 8>
gap> hh:=GeneratorsOfPresentation(P);
[ _x1, _x2, _x3 ]
gap> AddRelator(P,hh[2]);
gap> SimplifyPresentation(P);
#I there are 2 generators and 0 relators of total length 0
gap> # The free group presented by P is a homomorphic image
gap> # of a subgroup of index 120 in G7. Hence G7 is large.
gap> # QED.
```

Verification that G_{29} is large.

```
gap> F:=FreeGroup(["x","y"]);; x:=F.1;; y:=F.2;;
gap> U:=x*y;; V:=x*y^2;;
gap> W:=U^4*V^4*U^2*V^3*U*V^2*U^3*V^3*U^2*V;;
gap> G:=F/[x^2,y^3,W^2];;
gap> H:=Subgroup(G,[(G.1*G.2)^5]);;
gap> P:=PresentationNormalClosure(G,H);
<presentation with 11 gens and 30 rels of total length 550>
gap> gg:=GeneratorsOfPresentation(P);;
gap> for j in [2,4,6,8,9] do AddRelator(P,gg[j]); od;
gap> SimplifyPresentation(P);
#I there are 2 generators and 0 relators of total length 0
```

The kernel of a representation $G_{29} \rightarrow A_5$ has the free group of rank 2 as a homomorphic image.

Details of proof of Lemma 4.

```

gap> #
gap> # Proof that  $\langle x, y | x^3 = y^3 = xyxy^2xy^2x^2yxyx^2yx^2y^2 = 1 \rangle$ 
gap> # contains a nonabelian free subgroup, using the
gap> # Levai-Rosenberger-Souvignier representation
gap> #
gap> R:=UnivariatePolynomialRing(Rationals,"t");;
gap> t:=IndeterminatesOfPolynomialRing(R)[1];;
gap> a:=-3*t^4+8*t;;
gap> b:=-4*t^4+11*t;;
gap> c:=2*t^3-6;;
gap> d:=-5*t^5+14*t^2;;
gap> e:=-7*t^5+19*t^2;;
gap> f:=t^6-3*t^3+1;;
gap> x:=[[a,b,c],[0,0,1],[d,e,-a]];;
gap> y:=[[d,e,-a],[3*(b*t-d),-d,-c*t],[1,0,0]];;
gap> #
gap> # Representation modulo f - ie
gap> #      G' -> GL(V)=GL(3,K) with |K,Q|=6.
gap> #
gap> # Next check the relators
gap> #
gap> x^3 mod f;
[[ 1, 0, 0 ], [ 0, 1, 0 ], [ 0, 0, 1 ]]
gap> y^3 mod f;
[[ 1, 0, 0 ], [ 0, 1, 0 ], [ 0, 0, 1 ]]
gap> x*y*x*y^2*x*y^2*x^2*y*x*y*x^2*y*x^2*y^2 mod f;
[[ 1, 0, 0 ], [ 0, 1, 0 ], [ 0, 0, 1 ]]
gap> Id := last;      # identity matrix
[[ 1, 0, 0 ], [ 0, 1, 0 ], [ 0, 0, 1 ]]
gap> m:=x*y mod f;
[[ -2, -1, -t^5+3*t^2 ], [ 1, 0, 0 ], [ 0, 0, 1 ]]
gap> # Clearly det(m)=1 and tr(m)= -1
gap> # in fact m has eigenvalues +1, -1, -1:
gap> Rank(m-Id);
2
gap> Rank(m+Id);
2
gap> # and the -1 eigenspace is 1-dimensional

```

```

gap> # hence m^2 is nontrivial parabolic
gap> # (this is the L-R-S proof that the group is infinite)
gap> # Next find eigenvectors
gap> ev1:=[[1],[1],[4*t]];;
gap> ev2:=[[1],[-1],[0]];;
gap> m*ev1 mod f;
[ [ 1 ], [ 1 ], [ 4*t ] ]
gap> m*ev2 mod f;
[ [ -1 ], [ 1 ], [ 0 ] ]
gap> # Defining equation for the span P of the eigenvectors
gap> q:=[2*t,2*t,-1]];;
gap> q*ev1;
[ 0 ]
gap> q*ev2;
[ 0 ]
gap> # now construct a conjugate of m
gap> n:=y*x mod f;;
gap> # n also has eigenvalues +1,-1,-1, with 1-dimensional
gap> # (-1)-eigenspace we check that the plane P' spanned by
gap> # its eigenvectors does not coincide with the span P of the
gap> # eigenvectors of m. To do this, show that P is not
gap> # invariant under n
gap> q*n*ev1 mod f;
[ t^4+t ]
gap> q*n*ev2 mod f;
[ t^4-t ]
gap> # Thus P and P' are distinct, so intersect in a line L.
gap> # m^2 and n^2 fix L. They act on the quotient V/L as parabolics
gap> # with distinct fixed spaces, so they generate a
gap> # non-elementary subgroup of PSL(2,C). QED.

```

APPENDIX C: PICTURES

In this appendix we recall the theory of pictures over a free product of groups, and use it to prove Theorem 6.

Suppose that Γ_1, Γ_2 are groups, and $U \in \Gamma_1 * \Gamma_2$ is a cyclically reduced word of length at least 2. (Here and throughout this appendix, *length* means length in the free product sense.) A word $V \in \Gamma_1 * \Gamma_2$ is called a *piece* if there are words V', V'' with $V' \neq V''$, such that each of $V \cdot V', V \cdot V''$ is cyclically reduced as written, and each is equal to a cyclic conjugate of U or of U^{-1} . A cyclic subword of U is a *non-piece* if it is not a piece.

By a *one-relator product* $(\Gamma_1 * \Gamma_2)/U$ of groups Γ_1, Γ_2 we mean the quotient of their free product $\Gamma_1 * \Gamma_2$ by the normal closure of a cyclically reduced word U of positive length. Recall

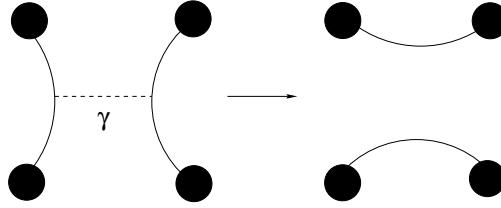


FIGURE 3. A bridge move

[3] that a *picture* over the one-relator product $G = (\Gamma_1 * \Gamma_2)/U$ is a graph \mathcal{P} on a surface Σ (which for our purposes will always be a disc) whose corners are labelled by elements of $\Gamma_1 \cup \Gamma_2$, such that

- (1) the label around any vertex, read in clockwise order, spells out a cyclic permutation of U or U^{-1} ;
- (2) the labels in any region of $\Sigma \setminus \mathcal{P}$ either all belong to Γ_1 or all belong to Γ_2 ;
- (3) if a region has k boundary components labelled by words $W_1, \dots, W_k \in \Gamma_i$ (read in anti-clockwise order; with $i = 1, 2$), then the quadratic equation

$$\prod_{j=1}^k X_j W_j X_j^{-1} = 1$$

is solvable for X_1, \dots, X_k in Γ_i . (In particular, if $k = 1$ then $W_1 = 1$ in Γ_i).

Note that edges of \mathcal{P} may join vertices to vertices, or vertices to the boundary $\partial\Sigma$, or $\partial\Sigma$ to itself, or may be simple closed curves disjoint from the rest of \mathcal{P} and from $\partial\Sigma$.

Pictures may be modified using *bridge moves* (Figure 3), defined as follows. Let γ denote an arc in the surface Σ , that meets the picture \mathcal{P} only in its endpoints, which are interior points of arcs of Γ . A bridge move is the result of altering Γ by surgery along γ . It is allowed provided that the resulting picture satisfies the above rules concerning labels within a region. The only example of this that we will use in practice is that γ divides a simply-connected region (say a Γ_1 -region) into two parts. The requirement for a bridge move is then that, in the resulting subdivision of the region-label into two subwords, each of the two subwords represent the identity in Γ_1 .

The *boundary label* of \mathcal{P} is the product of the labels around $\partial\Sigma$. By a version of van Kampen's Lemma, there is a picture on the disc with boundary label $W \in \Gamma_1 * \Gamma_2$ if and only if W belongs to the normal closure of U .

A picture is *minimal* if it has the fewest possible vertices among all pictures with the same (or conjugate) boundary labels. In particular every minimal picture is *reduced*: no edge e joins two distinct vertices in such a way that the labels of these two vertices that start and finish at the endpoints of e are mutually inverse.

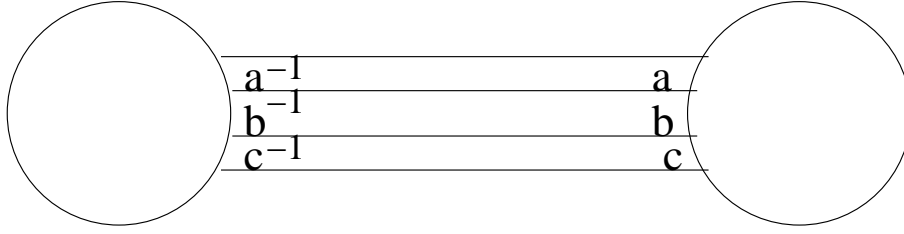


FIGURE 4. A piece

In a reduced picture, any collection of parallel edges between two vertices (or from one vertex to itself) corresponds to a collection of consecutive 2-gonal regions, and the labels within these 2-gonal regions spell out a piece (Figure 4).

Since U is cyclically reduced, no corner of an interior vertex is contained in a 1-gonal region.

Recall the statement of Theorem 6:

Theorem 6 *Let $N \gg 1$ be an integer. Then $A := (xy)^N$ and $B := (xy^{-1})^N$ freely generate a free subgroup of G .*

To prove the theorem, we suppose the conclusion false and derive a contradiction. There must be a minimal disc-picture \mathcal{P}_1 with boundary label a cyclically reduced word in $\{A, B\}$ (rewritten as a cyclically reduced word Z_1 in $\mathbb{Z}_2 * \mathbb{Z}_3$).

Note that there is very limited cancellation possible when re-writing a cyclically reduced word in $\{A, B\}$ as a cyclically reduced word in $\mathbb{Z}_2 * \mathbb{Z}_3$. In particular the word Z_1 has no cyclic subword of the form $(y(xy^2)^n xy)^{\pm 1}$ for $2 \leq n \leq N - 3$.

To simplify our analysis of pictures, we note that we can apply bridge-moves to make every \mathbb{Z}_2 -region a 2-gon. We then suppress the \mathbb{Z}_2 -regions, replacing each by a single arc.

We will also make the following assumption: among all minimal pictures with boundary label $Z(A, B)$ (and with \mathbb{Z}_2 -regions suppressed as above, we have chosen \mathcal{P} to maximise the number of triangular \mathbb{Z}_3 -regions).

We will further simplify our picture by amalgamating each maximal collection of parallel arcs into a single *edge* e . We say that the piece P defined by the collection of parallel arcs represented by e at a vertex v is *carried* by e at v . The corresponding piece at the other end of e will be called the *match* of P .

Up to cyclic permutation in $\mathbb{Z}_2 * \mathbb{Z}_3$ we can write

$$W = W_{13} = xy^{\alpha(0)}x \cdots xy^{\alpha(C)},$$

where the αi are indexed hexadecimally $0, 1, \dots, C$ such that $\alpha(0) = \alpha(2) = \alpha(3) = \alpha(4) = \alpha(5) = \alpha(9) = \alpha(A) = +1$ and $\alpha(1) = \alpha(6) = \alpha(7) = \alpha(8) = \alpha(B) = \alpha(C) = -1$.

Without further comment we will interpret the indices modulo 13 (D in hexadecimal), and use the interval notation $[i..j]$ as shorthand for the cyclic subword $xy^{\alpha(i)}x \cdots xy^{\alpha(j)}$ of W . We will use the bar notation $\overline{[i..j]}$ for the inverse of $[i..j]$. Then the product

$$[0..4] \cdot [5..9] \cdot [A..1] \cdot [2..5] \cdot [6..A]$$

of five non-pieces represents a proper subword of W^2 .

It is also worth remarking that $[7..B]$ is a non-piece.

Lemma 10. *Let v be a vertex in a reduced picture over G that is not joined by arcs to the boundary. Then v has index at least 5. If v has index 5 then the corners (in cyclic order) have labels from the sets $\{2, 3, 4\}$, $\{7, 8, 9\}$, $\{0, 1\}$, $\{5\}$, $\{A\}$.*

Proof. This follows from the above remarks about non-pieces. Indeed from the decomposition of (a subword of) W^2 into 5 non-pieces, each corner label must belong to the intersection of two non-adjacent non-pieces of the decomposition, namely $[0..4] \cap [A..1] = \{0, 1\}$, $[5..9] \cap [2..5] = \{5\}$, $[A..1] \cap [6..A] = \{A\}$, $[2..5] \cap [0..4] = \{2, 3, 4\}$ and $[6..A] \cap [5..9] = \{6, 7, 8, 9\}$. The fact that $[7..B]$ is also a non-piece allows us to discard 6 as a possible corner label, giving the result. \square

We may apply the above result to our minimal picture \mathcal{P} . But we may also apply it to other reduced pictures; we exploit this in the proof of Lemma 14 below to obtain information about boundary vertices of \mathcal{P} .

Here is one more simple result which will be useful later.

Lemma 11. *If v is an interior vertex of degree 5, then no triangular corner of v can be corner 2, 3 or 9.*

Proof. By Lemma 10, we know that if 2 or 3 is a corner at v in a region R , then A is an adjacent corner at v . The edge between these corners carries a piece $[B..1]$ or $[B..2]$. In each case the unique match $\overline{[0..3]}$ or $\overline{[C..3]}$ extends to a match $\overline{[B..3]}$ for $[B..3]$. So the other end u of e has the same orientation as v , and corner A in R . In particular R has two consecutive corners with opposite labels. This is not possible if R is triangular.

A similar argument applies if v has a corner 9 in R . There is an adjacent corner 0 or 1 at v , the edge between these corners carries $[A..C]$ or $[A..0]$ at v . Each of the possible matches, namely $[5..7]$, $\overline{[4..6]}$ and $\overline{[9..B]}$ for $[A..C]$ or $\overline{[8..B]}$ for $[A..0]$, extends to a match for $[9..C]$, or $[9..0]$ so the labels of the corners of R at u, v cancel, and hence R cannot be a triangular region. \square

C.1: Angles and Curvature. Assign angles $\theta(c)$ to corners c of \mathcal{P} as follows.

- (1) A corner of a boundary region between a boundary edge and a non-boundary edge has angle $\pi/2$.
- (2) A corner of a boundary region between two non-boundary edges has angle π .
- (3) A corner between a boundary edge and ∂D^2 has angle $\pi/2$.

- (4) A corner of a non-simply connected region has angle π .
- (5) If an interior vertex of degree 5 has precisely one non-triangular corner, then each such corner has angle $2\pi/3$.
- (6) If an interior vertex of degree 5 has two or more non-triangular corners, then each such corner has angle $\pi/2$.
- (7) All other corners have angle $\pi/3$.

The angle assignments lead to a measure of curvature for vertices and regions of \mathcal{P} , as follows.

- (1) The curvature $\kappa(v)$ of a vertex v is 2π less the sum of the angles of corners at v .
- (2) The curvature $\kappa(R)$ of an interior region R is $2\pi\chi(R) + \sum_c(\theta(c) - \pi)$, where χ denotes Euler characteristic, and the sum is over all corners c in R .
- (3) The curvature $\kappa(R)$ of a boundary region R is $\pi\chi(R) + \sum_c(\theta(c) - \pi)$, where χ denotes Euler characteristic, and the sum is over all corners c in R .

It is an immediate consequence of Euler's formula that the total curvature, summed over all the vertices and regions of \mathcal{P} , is $+2\pi$.

Theorem 12. *Let v be an interior vertex of positive κ -curvature: $\kappa(v) > 0$. Then the sequence of non-2-gonal corner labels (in cyclic order) around v is $5, A, 4, 7, 1$. Moreover the pieces $[2..4]$, $[6..9]$ at v are matched to $[3..5]$ and $[1..4]$ respectively at the corresponding neighbouring vertices.*

Proof. By Lemmas 10 and 11 the vertex v must have index 5 and has three consecutive corners $5, A, 4$, with the fourth and fifth corners being respectively 7 or 8 and 0 or 1. Since $\kappa(v) > 0$ it follows that all five of the corners incident at v belong to triangular regions. In a triangular region the corner labels are all equal (to y or to y^{-1}). This allows us to eliminate most of the possible combinations of corner labels given in Lemma 10.

If the fourth corner of v is 8, then the edge between the third and fourth corners of v carries $[5..7]$. Each possible match $[A..C]$, $[4..6]$, $[9..B]$ extends to a match for $[4..7]$, so the third corner cannot be triangular. This contradiction shows that the fourth corner of v must be 7.

A similar argument applies if the fifth corner is 0. The edge between the fourth and fifth corners carries $[8..C]$. The unique match $[9..0]$ for this extends to a match $[8..0]$ for $[8..0]$, so the fifth corner is not triangular.

This shows that the corners of v are $5, A, 4, 7, 1$, as claimed.

We must now show that the matches for the pieces $[2..4]$ and $[6..9]$ are as claimed.

Consider the piece $[2..4]$, represented by the group of arcs between the fifth and first corners. The two possible matches for it are $[3..5]$ and $[6..8]$. But the latter extends to a match $[6..9]$ for $[1..4]$. Since the fifth corner is triangular, the match must be $[3..5]$.

The piece $[6..9]$, represented by the group of arcs between the first and second corners, has only one possible match, namely $[1..4]$.

□

We next show that positive curvature does not arise in regions.

Proposition 13. *Let Δ be a region. Then $\kappa(\Delta) \leq 0$.*

Proof. Since no angle is greater than π , the result follows immediately for regions of non-positive Euler characteristic, so it suffices to consider regions which are topological discs.

If Δ is a boundary region which is a disc, then it has at least 4 corners with angle $\pi/2$ (and hence $\kappa(\Delta) \leq 0$), except in one of the following two possible cases.

- (1) $\Delta = D^2$. In this case \mathcal{P} is empty, and Z is the empty word, contrary to hypothesis.
- (2) $\partial\Delta$ consists of a single edge e together with a single arc $\gamma \subset \partial D^2$. In this case the label on γ is trivial, so e and Δ can be removed from \mathcal{P} , contrary to the assumption of minimality.

By definition, if Δ is a triangle, then $\theta(c) = \pi/3$ for every corner c of Δ , and so $\kappa(\Delta) = 0$.

Hence we may assume that Δ is an interior k -gonal region for some $k \geq 4$. By definition $\theta(c) \leq 2\pi/3$ for each corner c of Δ , and so if $k \geq 6$ then $\kappa(\Delta) \leq 0$. Hence we are reduced to the case where $k \in \{4, 5\}$.

In any 5-gonal interior region, the labels on the corners must consist of 4 labels y and one y^2 (or *vice versa*), and so a bridge-move is possible across such a region (in either of two ways).

A 5-gon with $\kappa > 0$ must have all its vertices of degree 5. However, any edge joining interior vertices of degree 5 represents two or more arcs. So a bridge-move across such a region creates in its place a triangular region and a 4-gonal region, leaving the rest of the picture unchanged.

It follows from our assumption of maximality of the number of triangles that no pentagonal region has $\kappa > 0$. Thus we are reduced to the case $k = 4$. In particular, precisely two of the corner labels in Δ are y and two are y^2 .

Note also that $\kappa(\Delta) \leq 0$ unless Δ has at least three corners belonging to vertices of degree 5. By Lemma 10 the corner of any such vertex in Δ has label one of 0, 1, 2, 3, 4, 5, 7, 8, 9, A .

We complete the proof by considering separate cases.

Case 1. Δ has a vertex of degree 5 with corner 5 in Δ .

We assume without loss of generality that this vertex v has positive orientation.

The anti-clockwise edge of Δ at v carries $[6..9]$ which has a unique match $\overline{[1..4]}$, so the neighbouring vertex at the other end of this edge has positive orientation and corner 5 (hence label y) in Δ .

The clockwise edge from v in Δ carries $[n..4]$ with $n \in \{1, 2\}$. The possible matches are $\overline{[6..9]}$ or $\overline{[6..8]}$ (if the next vertex is positively oriented); and $\overline{[2..5]}$ (if it is negatively oriented). In either case the next vertex has corner in Δ labelled y (positively oriented corner 0 or negatively oriented corner 6). But then the 4-gon Δ has at least three corners labelled y , which is impossible (Figure 5).

This contradiction shows that the result holds whenever Δ has a vertex of degree 5 with 5-corner in Δ .

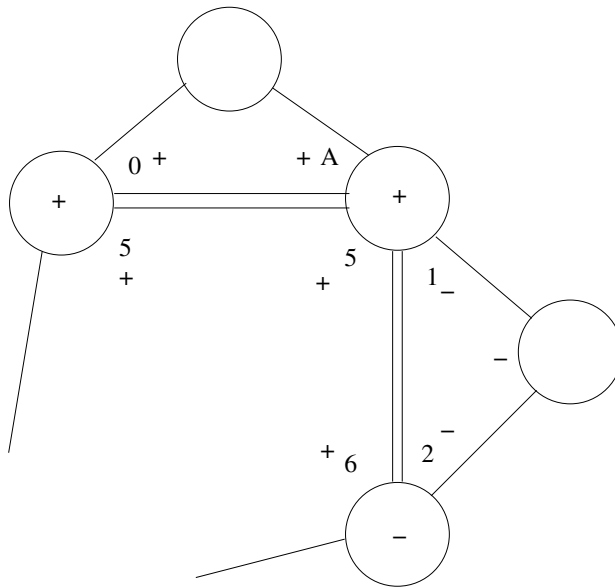


FIGURE 5. Case 1

As in Case 1, we assume without loss of generality that this vertex v has positive orientation.

The anti-clockwise edge from v in Δ carries $[B..n]$ with $n \in \{1, 2, 3\}$. In each case there is a unique match $[(1-n)..3]$. Hence the neighbouring vertex at the other end of this edge has positive orientation and corner 4 (hence label y) in Δ . Thus Δ has three corners labelled y , a contradiction (Figure 6).

Case 3. Δ has a vertex of degree 5 with corner 0 in Δ .

The anti-clockwise edge of Δ at v carries [1..4] which has a unique match $\overline{[6..9]}$, so the neighbouring vertex v_0 at the other end of this edge has positive orientation and corner A (hence label y) in Δ . By Case 2, we may assume that this vertex has degree > 5 , so the result follows unless the remaining two vertices of Δ have degree 5 and corner label y^2 in Δ .

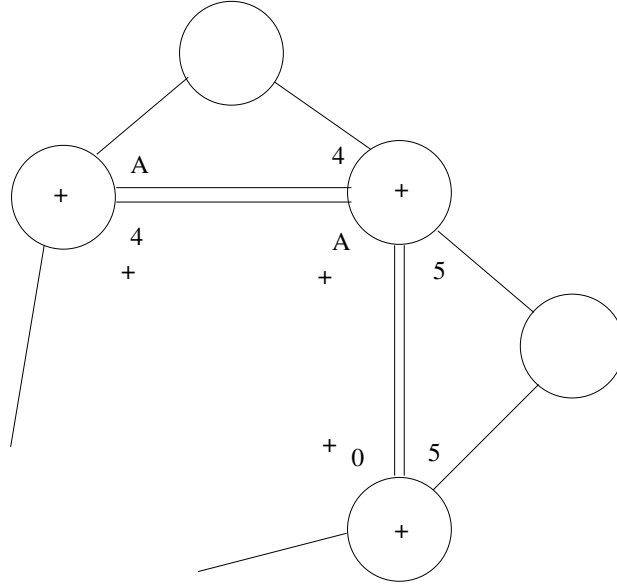


FIGURE 6. Case 2

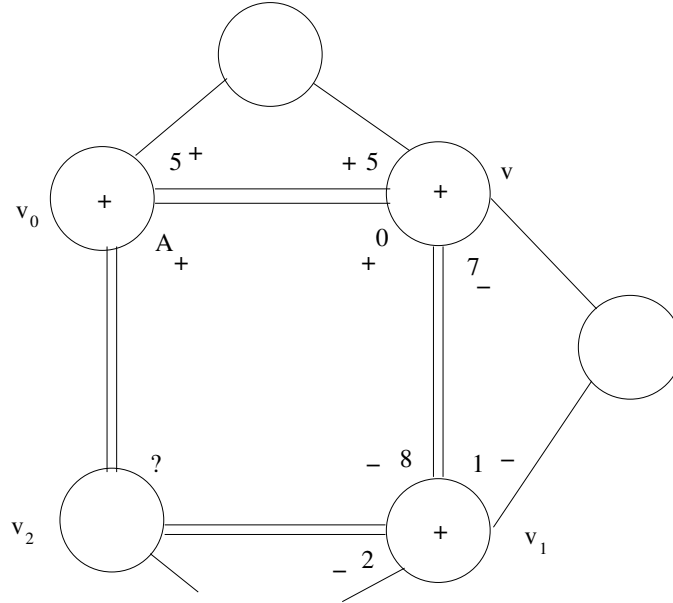


FIGURE 7. Case 3

The clockwise edge of Δ at v carries $[n..C]$ for $n \in \{8, 9, A\}$. Since v and its neighbour v_1 have opposite corner labels in Δ , the match must extend to a match for $[n..0]$. The only possibility is $\overline{[9..(8-n)]}$, so the vertex is positively oriented with corner 8 in Δ (Figure 7).

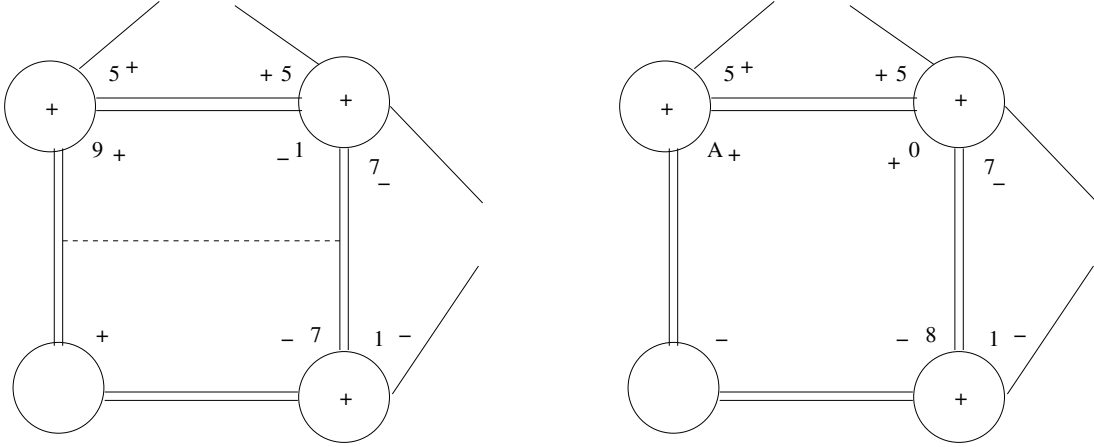


FIGURE 8. Bridge move in Case 4

Now consider the edge of Δ connecting v_1 to the fourth vertex v_2 of Δ . This edge carries $[n..7]$ at v_1 , for $n \in \{3, 4, 5\}$. The match at v_2 cannot extend to a match for $[n..8]$, since the corners in Δ of v_1 and v_2 have the same label. The only possibilities are $[(n+5)..C]$ and $[[9..(4-n)]]$.

Thus v_2 is either positively oriented with corner 8 in Δ , or negatively oriented with corner 0 in Δ . In the first case, we may apply the same argument to the edge joining v_2 to v_0 and deduce that v_0 has corner 0 or 8 in Δ . But we have already seen that v_0 has corner A in Δ , a contradiction. In the second case, we may apply the whole analysis to the corner v_2 rather than v , and deduce that v_0 and v_1 are negatively oriented. Again this is a contradiction.

This contradiction shows that the result holds whenever Δ has a vertex of degree 5 with 5-corner in Δ .

Case 4. Δ has a vertex of degree 5 with corner 1 in Δ .

As in Case 1, we assume without loss of generality that this vertex v has positive orientation.

The clockwise edge of Δ at v carries $[n..0]$ with $n \in \{8, 9, A\}$. There is a unique match $[[8..(8-n)]]$. Hence the neighbouring vertex v_1 at the other end of this edge has positive orientation and corner 7 in Δ . In particular the corners of v, v_1 in Δ both have the same label y^2 . Hence we may assume that the other two corner labels in Δ are y .

If any edge of Δ consists only of a single arc, then the vertices at either end of this edge each have degree > 5 , and so $\kappa(\Delta) \leq 0$. Hence we may assume that each edge of Δ consists of two or more arcs. Now carry out a bridge move across Δ , replacing Δ by a 4-gonal region Δ' with corner 0 at v (Figure 8). This bridge move has not changed the degree of any vertex of Δ , so $\kappa(\Delta) = \kappa(\Delta')$. But $\kappa(\Delta') \leq 0$ by Case 3.

Case 5. Δ has a vertex of degree 5 with corner 4 in Δ .

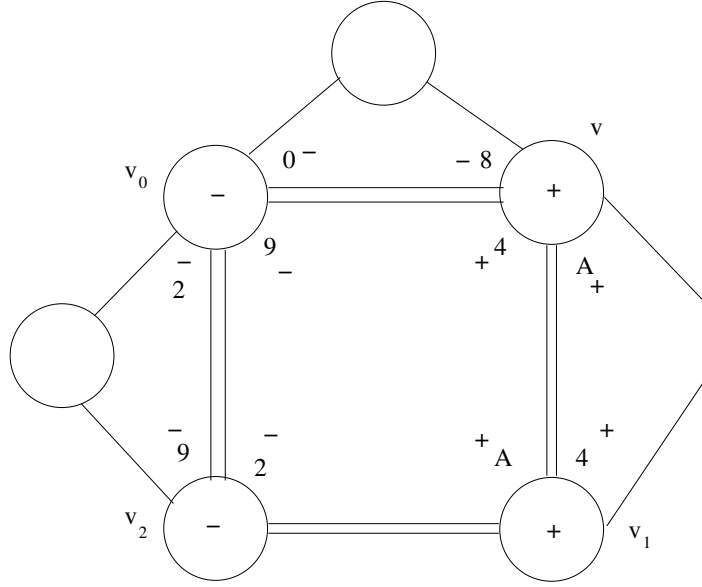


FIGURE 9. Case 5

As in Case 1, we assume without loss of generality that this vertex v has positive orientation.

The clockwise edge of Δ at v carries $[B..3]$. There is a unique match $\overline{[B..3]}$. Hence the neighbouring vertex v_1 at the other end of this edge has positive orientation and corner A in Δ . In particular we may assume by Case 2 that v_1 has degree > 5 . Moreover, the corners of Δ at v, v_1 are both labelled y , so we may assume that the other two corners are of degree 5 and labelled y^2 .

The anti-clockwise edge of Δ at v carries $[5..n]$ with $n \in \{6, 7, 8\}$. We may assume that this edge separates Δ from a triangular region, for otherwise each of its incident vertices v, v_0 has at least 2 non-triangular corners. This, together with the fact that v_1 has degree > 5 , would imply that $\kappa(\Delta) \leq 0$.

Since the labels of the corners of Δ at v, v_0 are opposite, the match for $[5..n]$ at v_0 must extend to a match for $[4..n]$, but not to a match for $[5..(n+1)]$. There are precisely three possible matches: $[A..C]$ or $\overline{[9..B]}$ with $n = 7$, and $\overline{[3..6]}$ with $n = 8$. Of these, we may rule out $\overline{[9..B]}$, since it would mean that v_0 had a corner C and so degree at least 6. Since neither 2 nor 9 can be a corner of a triangular region at an interior vertex of degree 5, we are left only with the possible match $[A..C]$. Thus v_0 is negatively oriented, with corner 9 in Δ , and label y^2 .

Now consider the edge from v_0 to the fourth vertex v_2 of Δ . Again we may assume that this edge separates Δ from a triangular region T . The corners at v_0, v_2 in T must be equal, as are the corners at v_0, v_2 in Δ . This edge carries $[n..8]$ at v_0 , and the match at v_2 must be

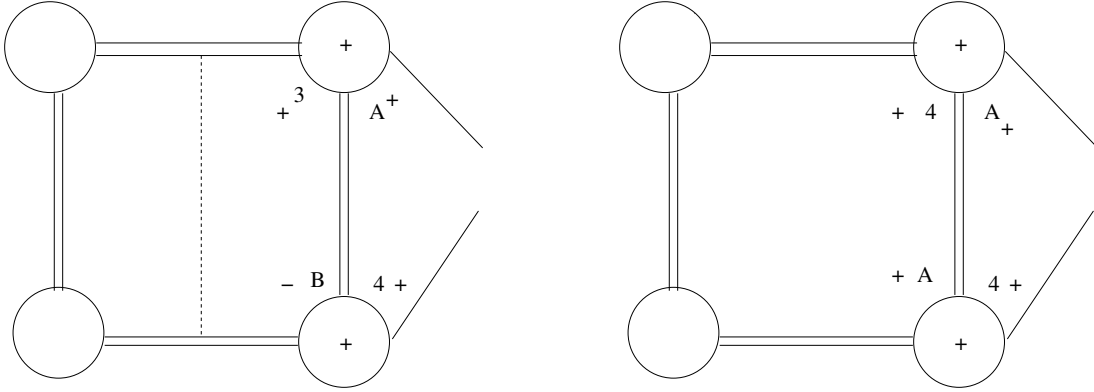


FIGURE 10. Bridge move in Case 6

maximal. The only possible match is $\overline{[3..8]}$ with $n = 3$. In particular v_0 has corner 2 in T . But an interior vertex of degree 5 cannot have a triangular corner 2, so once again we have a contradiction (Figure 9).

Case 6. Δ has a vertex of degree 5 with corner $c \in \{2, 3\}$ in Δ .

As in Case 1, we assume without loss of generality that this vertex v has positive orientation.

The clockwise edge of Δ at v carries $[B..(c-1)]$. There is a unique match $\overline{[(2-c)..3]}$. Hence the neighbouring vertex v_1 at the other end of this edge has positive orientation and corner B or C in Δ , and hence label y^2 . In particular v_1 has degree > 5 . Thus we may assume that the other two corners have degree 5.

Arguing as in Case 4, we may perform a bridge move across Δ without changing the degrees of the incident vertices or changing the rest of the picture (Figure 10). The resulting 4-gon Δ' has a corner $c+1$. If $c = 3$ then by Case 5 we have $\kappa(\Delta') \leq 0$. Hence also $\kappa(\Delta) = \kappa(\Delta') \leq 0$. If $c = 2$ then Δ' has a vertex of degree 5 with corner 3 in Δ' . As we have just seen, this implies that $\kappa(\Delta') \leq 0$, and hence by the same argument again we have $\kappa(\Delta) \leq 0$.

Conclusion

Using the results of Cases 1-6, we may assume that Δ has three consecutive vertices of degree 5, each with corner 7, 8 or 9 in Δ . We assume without loss of generality that the middle vertex v of this sequence has positive orientation, and corner $c \in \{7, 8, 9\}$.

Suppose first that $c = 9$. Then the clockwise edge of Δ from v carries $[n..8]$ for $n \in \{3, 4, 5\}$. There is a unique match $\overline{[3..(11-n)]}$ for this, so the vertex v_1 at the other end of this edge has corner 2 in Δ , contrary to hypothesis. Hence $c \in \{7, 8\}$.

The anti-clockwise edge of Δ at v carries $[(c+1)..n]$ with $n \in \{C, 0\}$. If $n = 0$ then there is a unique match $\overline{[8..(7-c)]}$. Hence the neighbouring vertex v_0 at the other end of this

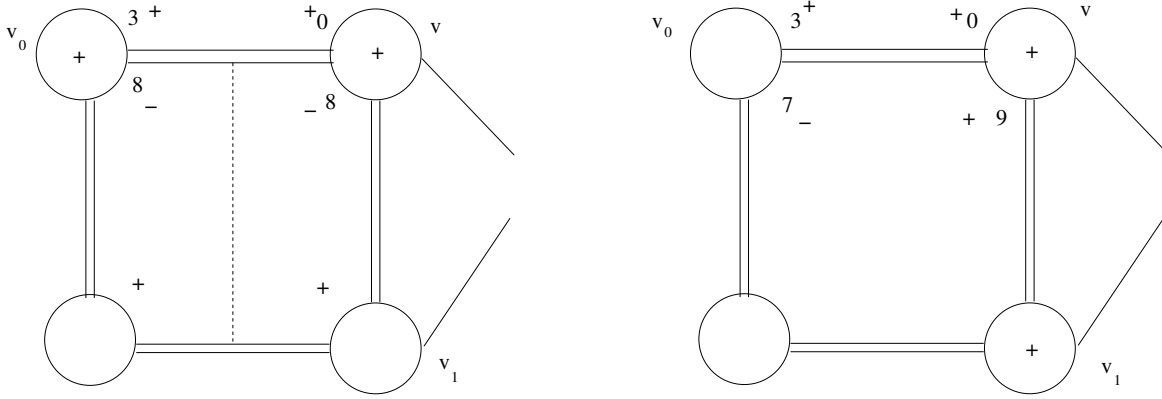


FIGURE 11. Bridge move to complete proof

edge has positive orientation and corner 0, 1 or C in Δ , contrary to assumption. So we may assume that $n = C$. If $c = 7$ then again there is a unique match $\overline{[9..0]}$; in this case v_0 has corner 1 in Δ , again giving a contradiction. So we may assume that $c = 8$. Possible matches for $[9..C]$ are $\overline{[9..C]}$, $\overline{[4..7]}$ and $\overline{[4..7]}$. The corner of v_0 in Δ is then 0, 3, or 8 respectively.

By hypothesis, v_0 has corner 7, 8 or 9 in Δ , so the only possible match is $\overline{[4..7]}$. In particular v_0 is positively oriented, with corner 8 in Δ and hence label y^2 . The corner of Δ at v also has label y^2 , so we may assume that the other two corners of Δ have label y .

As in previous cases, we may assume that each edge of Δ corresponds to two or more arcs, so that a bridge move across Δ has the effect of replacing Δ by another 4-gonal region Δ' , without changing the degrees of the vertices of Δ (Figure 11). Such a bridge-move also has the effect that the corners of v, v_0 in Δ change from 8 to 9, 7 respectively. Hence $\kappa(\Delta) = \kappa(\Delta') \leq 0$, as required.

This completes the proof. □

As a consequence of the above result, any positive curvature in our picture is concentrated at interior vertices of degree 5 with 5 triangular corners.

Hence there exist interior vertices with $\kappa > 0$. To complete the proof we distribute positive curvature from these vertices to negatively curved neighbours.

Recall from Theorem 12 that any such interior vertex v has corners 5, A , 4, 7, 1, and that the pieces $[2..4]$ and $[6..9]$ are matched at the neighbouring vertices u_0, u_1 by $[3..5]$ and $[1..4]$ respectively (Figure 12).

We say that these two vertices u_0, u_1 are *associated* to v . Note that each of u_0, u_1 is connected to v by an edge carrying a word that contains $[3..4]$. In particular, no vertex is associated to more than 2 such interior vertices with $\kappa > 0$. It also follows that no such vertex can itself be positively curved.

We define the *amended* curvature function on vertices by

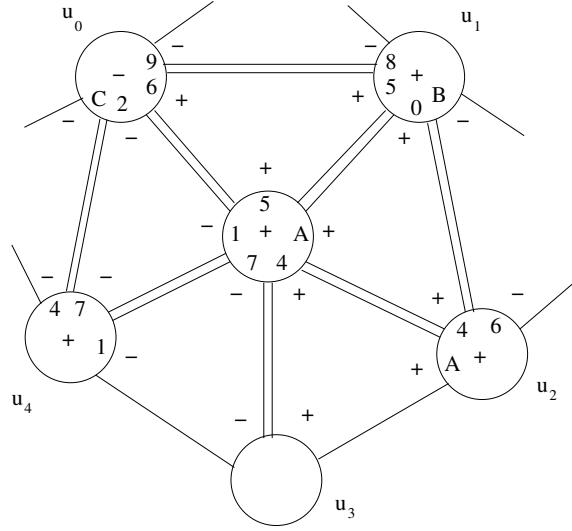


FIGURE 12. Neighbourhood of a positively curved vertex

- $\kappa'(v) = \kappa(v) + k\pi/6$ if u is associated to $k > 0$ interior vertices of positive (κ -)curvature;
- $\kappa'(v) = 0$ if v is an interior vertex with $\kappa(v) > 0$;
- $\kappa'(v) = \kappa(v)$ otherwise.

In other words, we distribute $\pi/6$ of positive curvature from any positively curved vertex to each of its two associated vertices. There is no adjustment to the curvature of regions, so the total curvature remains unchanged. We have already seen that there are no positively curved regions, so it remains to check that there are no vertices with positive amended curvature.

Lemma 14. *For every vertex u of \mathcal{P} , $\kappa'(u) \leq 0$.*

Proof. Suppose first that u is an interior vertex of \mathcal{P} . If $\kappa(u) > 0$ then by Theorem 12 we know that u has degree 5 with all incident corners triangular, so $\kappa(u) = \pi/3$. By definition, $\kappa'(u) = 0$.

So we may assume that $\kappa(u) \leq 0$.

If u is not associated to any interior vertex with positive κ -curvature, then by definition $\kappa'(u) = \kappa(u) \leq 0$.

If u is associated to an interior vertex v of positive curvature, then v has a configuration as in Figure 12 above, and u is one of the neighbouring vertices u_0, u_1 .

Note that the edges connecting u to u_2, u_4 carry $[B..3]$ and $[8..0]$, which have unique matches $[\overline{B..3}]$ and $[\overline{8..0}]$ respectively. Hence u_2, u_4 are positively oriented. Hence also the corners of the triangle uu_1u_2 at u_1 and u_2 are 0 and 4 respectively, while those of uu_4u_0 at u_4 and u_0 are 7 and 2 respectively.

The labels following corner 4 at u_2 , in clockwise order, are y, y^2, y^2, \dots , while those following 0 at u_1 in anti-clockwise order are y^2, y^2, y, \dots . It follows that the edge from u_1 to u_2 has at most two arcs, so the next corner of u_2 anticlockwise from the corner 0 in uu_1u_2 is B or C .

By a similar analysis, we see that the edge u_4u_0 has at most 3 arcs, so that the next corner of u_4 clockwise from the corner 2 in uu_4u_0 is C , 0 or 1.

A similar analysis again shows that the edge u_0u_1 has at most 3 arcs, so the next corner of u_0 anti-clockwise from the 6 corner in uu_0u_1 is 7, 8 or 9, while the next corner of u_1 clockwise from the 5 corner in uu_0u_1 is 6, 7 or 8.

Finally, since $[A..1]$, $[2..5]$ and $[6..A]$ are non-pieces, it follows that each of u_0, u_1 have at least three incident corners in addition to the four in Figure 12. In other words, each has degree at least 7, and hence $\kappa(u) \leq -\pi/3$. Since u is associated to at most two interior vertices of positive curvature, $\kappa'(u) \leq \kappa(u) + 2\pi/6 \leq 0$, as required.

Now suppose that u is a boundary vertex. If u is connected to the boundary by more than one parallel class of arcs, then it has at least four corners of angle $\pi/2$, and hence $\kappa(u) \leq 0$. If in addition it is associated to a positively curved interior vertex, then it has also at least two interior corners, with angles $\pi/3$. Hence $\kappa(u) \leq -2\pi/3$ so $\kappa'(u) \leq -2\pi/3 + 2\pi/6 < 0$.

If u is not connected to the boundary, then it has degree at least 5 and at least one corner with angle π , so $\kappa(u) \leq -\pi/3$ and again $\kappa'(u) \leq 0$.

Assume therefore that u is connected to the boundary by a single parallel class of arcs. The word U carried by that class has to be a cyclic subword of W^2 , and must match a cyclic subword of Z .

The maximal such matches are

- $[(xy)^4(xy^{-1})^3x]^{\pm 1}$,
- $[(xy^{-1})^3(xy)^2x]^{\pm 1}$,
- $[xyxy^{-1}x(yx)^4]^{\pm 1}$, and
- $[(xy^{-1})^2xyxy^{-1}x]^{\pm 1}$.

In particular U is either a piece or a product of two pieces. We may extend \mathcal{P} to a new reduced picture $\widehat{\mathcal{P}}$, in which u is no longer attached to the boundary by arcs, by joining either one (if U is a piece) or two new vertices to u where it joins $\partial\mathcal{P}$.

If u is associated to a positively curved interior vertex v of \mathcal{P} , then v is also a positively curved interior vertex of $\widehat{\mathcal{P}}$ to which u is associated. It follows that u has degree at least 7 in $\widehat{\mathcal{P}}$, and hence at least 5 in \mathcal{P} . It therefore has at least 4 interior corners, and two boundary corners, so $\kappa(u) \leq 2\pi - 2\pi/2 - 4\pi/3 = -\pi/3$, and hence $\kappa'(u) \leq 0$ as required.

Otherwise $\kappa'(u) = \kappa(u)$ so it suffices to show that $\kappa(u) \leq 0$.

By Lemma 10, u has index at least 5 in $\widehat{\mathcal{P}}$, so at least 4 in \mathcal{P} , except possibly in the case where U is not a piece and the index of u in $\widehat{\mathcal{P}}$ is precisely 5. In this case U is one of $[(xy)^4(xy^{-1})^kx]^{\pm 1}$ (with $0 \leq k \leq 3$), $[(xy^{-1})^3(xy)^2x]^{\pm 1}$, $[xyxy^{-1}x(yx)^4]^{\pm 1}$ or $[xy^{-1}x(yx)^4]^{\pm 1}$. In the first case, u would be a vertex of index 5 in $\widehat{\mathcal{P}}$, not joined to the boundary, with an incident corner C . In the second case u would have an incident corner B . In the last

two cases u would have an incident corner 6. In all cases we would have a contradiction to Lemma 10. Hence u has index at least 4 in \mathcal{P} in all cases.

But then u has at least three corners of angle $\geq \pi/3$ together with two boundary corners of angle $\pi/2$, so $\kappa'(u) = \kappa(u) \leq 0$ as claimed. \square

This concludes the proof of Theorem 6.

REFERENCES

- [1] J. O. Button, *Proving finitely presented groups are large by computer*, Exp. Math. **20** (2011), no. 2, 153–168.
- [2] The GAP Group, GAP – Groups, Algorithms, and Programming. (<http://www.gap-system.org>)
- [3] J. Howie, *The quotient of a free product of groups by a single high-powered relator. I. Pictures. Fifth and higher powers*, Proc. London Math. Soc. (3) **59** (1989), 507–540.
- [4] J. Howie, *Generalised triangle groups of type (3, 3, 2)*, Preprint 2010: arXiv:1012.2763
- [5] J. Howie, *Generalised triangle groups of type (3, q, 2)*, Algebra and Discrete Mathematics **15** (2013), no. 1, 1–18.
- [6] J. Howie and A. G. T. Williams, *Free subgroups in certain generalized triangle groups of type (2, m, 2)*, Geom. Dedicata **119** (2006), 181–197.
- [7] L. Lévai, G. Rosenberger and B. Souvignier, *All finite generalized triangle groups*, Trans. Amer. Math. Soc. **347** (1995), 3625–3627.
- [8] G. Rosenberger, *On free subgroups of generalized triangle groups*, Algebra i Logika **28** (1989), 227–240, 245.
- [9] A. G. T. Williams, *Studies on generalised triangle groups*, Ph. D. Thesis, Heriot-Watt University (2000).

JAMES HOWIE, DEPARTMENT OF MATHEMATICS AND MAXWELL INSTITUTE FOR MATHEMATICAL SCIENCES, HERIOT-WATT UNIVERSITY, EDINBURGH EH14 4AS

E-mail address: J.Howie@hw.ac.uk

ALEXANDER KONOVALOV, SCHOOL OF COMPUTER SCIENCE, UNIVERSITY OF ST ANDREWS, NORTH HAUGH, ST ANDREWS KY16 9SX

E-mail address: alexander.konovalov@st-andrews.ac.uk